

THE TWO MEMBRANES PROBLEM FOR DIFFERENT OPERATORS

L. CAFFARELLI, D. DE SILVA, AND O. SAVIN

ABSTRACT. We study the two membranes problem for different operators, possibly nonlocal. We prove a general result about the Hölder continuity of the solutions and we develop a viscosity solution approach to this problem. Then we obtain $C^{1,\gamma}$ regularity of the solutions provided that the orders of the two operators are different. In the special case when one operator coincides with the fractional Laplacian, we obtain the optimal regularity and a characterization of the free boundary.

1. INTRODUCTION

In this paper we study the two membranes problem for different operators. Physically the problem consists in having two elastic membranes made of possibly different composite materials that are constrained one on top of the other. This is a double obstacle problem in which each membrane can be viewed as the obstacle for the other membrane, and the two obstacles interact at the same time.

The two membranes problem for the Laplacian was first considered by Vergara-Caffarelli [VC] in the context of variational inequalities. In this case the situation can be reduced to the classical obstacle problem by looking at the vertical distance between the membranes. The two membranes problem for a nonlinear operator was studied by Silvestre [S1]. He obtained the optimal $C^{1,1}$ regularity of the solutions together with a characterization of the regularity of the free boundary of the coincidence set. The key step is to show that the difference between the two solutions solves an obstacle problem for the linearized operator.

We also mention that a more general version of the two membranes problem involving N membranes was considered by several authors (see for example [ARS, CCVC, CV]).

The two membranes problem for different operators is more challenging mathematically. In the unconstrained parts the membranes solve different equations and therefore their difference solves a fourth order equation rather than a second order equation. For example even in the simplest case of two dimensions and two linear operators, say Δ and $\tilde{\Delta} := \partial_{xx} + 2\partial_{yy}$, the optimal regularity of the solutions seems to be a difficult problem.

In this paper we consider the two membranes problem for the large class of elliptic operators, possibly nonlocal, of order $2s \in (0, 2]$. The interest in the nonlocal case comes from the applications. It is well known for example that the classical Signorini problem in elasticity which consists in finding the equilibrium position of an elastic body resting on a rigid surface, is modeled by an obstacle problem for the fractional Laplacian $\Delta^{1/2}$. In the case when the elastic body presses against a membrane, one

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obtains a two membranes obstacle problem involving a fractional Laplacian and a second order operator.

In the general case, we prove a result about the Hölder continuity of the solutions and we develop a viscosity solution approach. Then we obtain better regularity properties of the solutions provided that the orders of the two operators are different. Heuristically this situation corresponds to the case when one membrane, say the lower membrane, is more sensitive to small infinitesimal changes. From this we can already deduce a certain initial regularity of the lower membrane. Then, the regularity of the upper membrane can be obtained by solving the obstacle problem in which the obstacle is given by the lower membrane. In order to obtain the optimal regularity we need to repeat these arguments several times. A large part of the paper is devoted to obtaining estimates for various obstacle problems which are optimal with respect to the smoothness of the obstacle. We first discuss the general case of operators that correspond to translation invariant kernels. Then we consider the special case of the fractional Laplacian. As mentioned above in the course of the paper we also treat the obstacle problem for translation invariant kernels which is of independent interest.

The paper is organized as follows. In Section 2 we formulate the two membranes problem and state precisely our results. In Section 3 we obtain the Hölder regularity of the minimizing pair. In Section 4 we develop the viscosity approach to the two membranes problem. In Section 5 we deal with the translation invariant kernels and finally in Section 6 we discuss the case of the fractional Laplacian. The Appendix is devoted to the proof of Schauder estimates for nonlocal equations.

2. MAIN RESULTS

2.1. Notation. Let $s \in (0, 1)$ and let $k(x, y)$ be a symmetric, measurable kernel proportional to $|x - y|^{-n-2s}$, i.e.

$$0 < \lambda \leq k(x, y)|x - y|^{n+2s} \leq \Lambda, \quad k(x, y) = k(y, x).$$

Given a function $u \in L^2_{loc}$ we define its H^s seminorm in B_1 , the unit ball, as

$$\|u\|_{H^s(B_1)}^2 := \frac{1}{2} \int \int_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (CB_1 \times CB_1)} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy,$$

and if $\|u\|_{H^s(B_1)} < \infty$ we write $u \in H^s(B_1)$. Here for any set $E \subset \mathbb{R}^n$, we denote by $\mathcal{C}E$ its complement in \mathbb{R}^n .

It is not difficult to check that

$$(2.1) \quad \left\| u - \int_{B_1} u \right\|_{L^2(\mathbb{R}^n, d\omega)} \leq C \|u\|_{H^s(B_1)}, \quad d\omega := \frac{dx}{1 + |x|^{n+2s}}.$$

Given two functions $u, v \in H^s(B_1)$ we define the “inner product” of u and v with respect to the kernel k as

$$(2.2) \quad \mathcal{E}_k(u, v) := \frac{1}{2} \int \int_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (CB_1 \times CB_1)} (u(x) - u(y))(v(x) - v(y)) k(x, y) dx dy.$$

If u minimizes the energy $\mathcal{E}_k(u, u)$ among all functions $u \in H^s(B_1)$ which are fixed outside B_1 , say $u = u^0 \in H^s(B_1)$ outside B_1 , then

$$\mathcal{E}_k(u, \varphi) = 0, \quad \forall \varphi \in H^s(B_1), \quad \text{with } \varphi = 0 \text{ outside } B_1.$$

The last equality can be written in the sense of distributions as $\mathcal{L}_k u = 0$ in B_1 , with

$$\langle \mathcal{L}_k u, \varphi \rangle := -\mathcal{E}_k(u, \varphi) \quad \forall \varphi \in C_0^\infty(B_1),$$

and formally $\mathcal{L}_k u$ can be written as the non-local operator

$$\mathcal{L}_k u(x) = \int (u(y) - u(x))k(y, x)dy.$$

We wish to include the case when k has order $s = 1$. In this case the quadratic form $\mathcal{E}_k(u, u)$ is given by

$$(2.3) \quad \mathcal{E}_A(u, u) = \int_{B_1} (\nabla u)^T A(x) \nabla u \, dx,$$

with $A(x)$ a symmetric $n \times n$ matrix satisfying $\lambda I \leq A(x) \leq \Lambda I$, and the linear operator associated to \mathcal{E}_A is

$$\mathcal{L}_A(u) = \operatorname{div}(A(x)\nabla u).$$

Finally, we notice the following scaling property of \mathcal{E}_k after space dilation. Let

$$\tilde{u}(x) = u(rx),$$

be the $1/r$ dilation of u in the space variable. Then

$$\mathcal{E}_k(u, v) = r^{n-2s} \mathcal{E}_{\tilde{k}}(\tilde{u}, \tilde{v})$$

where in the double integral on the right we remove the contribution coming from $\mathcal{C}B_{1/r} \times \mathcal{C}B_{1/r}$ and the kernel $\tilde{k}(x, y) := r^{n+2s}k(rx, ry)$ is the rescaling on k , and therefore satisfies the same growth conditions as k .

2.2. The two membranes problem – General case, Hölder continuity of the minimizers. We consider the two membranes obstacle problem in B_1 for operators corresponding to two different kernels k_1 and k_2 as above, with the order s_1 not necessarily equal to s_2 . We look for a pair of functions (u_1, u_2) , with $u_2 \leq u_1$ in B_1 and u_1, u_2 prescribed outside B_1 , which minimizes the energy functional

$$(2.4) \quad \mathcal{F}(u_1, u_2) := \mathcal{E}_{k_1}(u_1, u_1) + \mathcal{E}_{k_2}(u_2, u_2) + \int_{B_1} u_1 f_1 + u_2 f_2 \, dx,$$

among all $(u_1, u_2) \in \mathcal{A}$.

Here $f_i \in L^2(B_1)$ and \mathcal{A} represents the set of admissible pairs,

$$\mathcal{A} = \{(u_1, u_2) \mid u_2 \leq u_1, \quad u_i \in H^{s_i}(B_1), \quad u_i = u_i^0 \text{ outside } B_1\},$$

with $u_i^0 \in H^{s_i}(B_1)$, $u_2^0 \leq u_1^0$ in B_1 , a given pair of functions.

With the convention in the Subsection above, we allow in the definition of the energy \mathcal{F} also the cases when either one or both of the s_i 's equal to 1, and we need to replace the quadratic form accordingly.

Since \mathcal{F} is strictly convex, and $\mathcal{F}(u_1^0, u_2^0) < \infty$, we obtain the existence and uniqueness of a minimizing pair (u_1, u_2) by the standard methods of the calculus of variations.

Proposition 2.1. *There exists a unique minimizing pair $(u_1, u_2) \in \mathcal{A}$ for the functional \mathcal{F} in (2.4). Moreover $u_i \in L^2(\mathbb{R}^n, d\omega_i)$ and $\sum_i \|u_i\|_{L^2(d\omega_i)} \leq C$ for a constant C depending on the boundary data u_i^0 and on the f_i 's.*

We observe that to prove the L^2 bound for the minimizing pair, one uses (2.1). Notice that if $\varphi \geq 0$ and $\varphi \in C_0^\infty(B_1)$ then

$$(u_1 + \epsilon\varphi, u_2) \in \mathcal{A} \quad \text{and} \quad (u_1, u_2 - \epsilon\varphi) \in \mathcal{A},$$

which gives

$$(2.5) \quad \mathcal{L}_{k_1} u_1 \leq f_1, \quad \mathcal{L}_{k_2} u_2 \geq f_2 \quad \text{in } B_1,$$

in the sense of distributions, thus $\mathcal{L}_{k_1} u_1, \mathcal{L}_{k_2} u_2$ are Radon measures.

Moreover, if $\varphi \in C_0^\infty(B_1)$ is not necessarily positive we still have

$$(u_1 + \epsilon\varphi, u_2 + \epsilon\varphi) \in \mathcal{A},$$

hence

$$(2.6) \quad \mathcal{L}_{k_1} u_1 + \mathcal{L}_{k_2} u_2 = f_1 + f_2 \quad \text{in } B_1.$$

Equations (2.5)-(2.6) together with the inequality $u_2 \leq u_1$, can be viewed as the Euler-Lagrange characterization of the minimizing pair.

In this paper we are concerned with the regularity of the minimizing pair (u_1, u_2) and some properties of the free boundary Γ which is defined as the boundary of the coincidence set, i.e.

$$\Gamma := \partial\{u_1 = u_2\} \cap B_1.$$

Our first result is the following interior Hölder regularity of the minimizing pair.

Theorem 2.2. *Assume $f_i \in L^{q_i}(B_1)$ with $q_i > \frac{n}{2s_i}$. Let (u_1, u_2) be a minimizing pair. Then $u_i \in C^\alpha(B_1)$ and*

$$\sum_i \|u_i\|_{C^\alpha(B_{1/2})} \leq C \sum_i (\|u_i\|_{L^2(d\omega_i)} + \|f_i\|_{L^{q_i}(B_1)}),$$

with α and C depending on $n, \lambda, \Lambda, s_i, q_i$.

To obtain better regularity properties of the minimizing pair we need to require that the kernels k_i are more regular, as in the next subsection.

2.3. Translation invariant kernels – Viscosity solutions and higher regularity. We consider the case when k is translation invariant, i.e.

$$k(x, y) = K(x - y), \quad K(y) = K(-y),$$

and satisfies the natural growth condition of the gradient

$$|\nabla K(y)| \leq \frac{\Lambda}{|y|^{n+1+2s}}.$$

The integro-differential operator associated to this kernel can be written as

$$\mathcal{L}_K w(x) := PV \int_{\mathbb{R}^n} (w(y) - w(x)) K(y - x) dy.$$

and the value $\mathcal{L}_K w(x)$ is well-defined as long as $w \in L^1(\mathbb{R}^n, d\omega)$ and w is $C^{2s+\epsilon}$ at x .

In this case we show that the minimizing pair (u_1, u_2) satisfies

$$(2.7) \quad u_1 \geq u_2, \quad \mathcal{L}_{K_1} u_1 \leq f_1, \quad \mathcal{L}_{K_2} u_2 \geq f_2 \quad \text{in } B_1,$$

$$(2.8) \quad \mathcal{L}_{K_i} u_i = f_i \quad \text{on } \{u_2 > u_1\}, \quad \sum_i \mathcal{L}_{K_i} u_i = \sum_i f_i \quad \text{in } B_1,$$

in the viscosity sense, and moreover these inequalities determine uniquely the pair (u_1, u_2) (see Proposition 4.9).

When the orders of the operators \mathcal{L}_{K_i} are different we improve the result of Theorem 2.2 and obtain the $C^{1,\gamma}$ regularity of the pair (u_1, u_2) . Notice that the two membranes may interact, that is $\{u_1 = u_2\} \cap B_1 \neq \emptyset$ independently of the sign of f_1, f_2 . We obtain the following result.

Theorem 2.3. *Assume $s_1 < s_2$ and u_i satisfy (2.7)-(2.8) with $f_i \in C^{0,1}(B_1)$. Then $u_i \in C^{\alpha_i}(B_1)$ with $\alpha_i > 1$,*

$$\alpha_1 = \max\{1, 2s_1\} + \epsilon_0, \quad \alpha_2 = \alpha_1 + 2(s_2 - s_1)$$

and

$$\sum_i \|u_i\|_{C^{\alpha_i}(B_{1/2})} \leq C \sum_i (\|u_i\|_{L^1(d\omega_i)} + \|f_i\|_{C^{0,1}(B_1)}),$$

with ϵ_0 and C depending on n, λ, Λ, s_i .

2.4. The obstacle problem for operators with translation invariant kernels. In order to obtain Theorem 2.3 we study the obstacle problem for the operator \mathcal{L}_K associated to a translation invariant kernel of order $2s$. We obtain the following result, of independent interest. Assume that u, φ are continuous in B_1 , $u \in L^1(\mathbb{R}^n, d\omega)$, and

$$(2.9) \quad u \geq \varphi \quad \text{in } B_1,$$

$$(2.10) \quad \mathcal{L}_K u \leq f \quad \text{in } B_1, \quad \text{and} \quad \mathcal{L}_K u = f \quad \text{in } \{u > \varphi\} \cap B_1,$$

with K of order $2s$ as at the beginning of subsection 2.3.

Theorem 2.4. *Let u be a solution to (2.9), (2.10), and assume that*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{0,1}(B_1)} \leq 1.$$

for some $\beta \neq 2s$.

Then $u \in C^\alpha(B_1)$ for $\alpha = \min\{\beta, \max\{1, 2s\} + \epsilon_0\}$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C,$$

where ϵ_0 depends on n, λ, Λ, s , and the constant C may depend also on β .

2.5. Fractional laplacian – Optimal regularity and the geometry of the free boundary. In the special case when

$$K(y) = \frac{1}{|y|^{n+2s}}$$

the operator \mathcal{L}_K reduces to the fractional Laplacian Δ^s and we obtain the optimal regularity of the solution. As usual, we can characterize the points on the free boundary

$$\Gamma := \partial\{u = \varphi\} \cap B_1.$$

Precisely the set $\Sigma \subset \Gamma$ of *singular points* consists of those $y \in \Gamma$ such that

$$(u - \varphi)(x) = o(|x - y|^{1+s}),$$

and $\Gamma \setminus \Sigma$ is the set of *regular points* (or stable points) of the free boundary.

Theorem 2.5. *Let u be a solution to (2.9), (2.10), with*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{\beta-2s}(B_1)} \leq 1, \quad \text{for some } \beta > 1 + s.$$

Then $u \in C^{1+s}(B_1)$ and

$$\|u\|_{C^{1+s}(B_{1/2})} \leq C.$$

Moreover, the free boundary Γ is a $C^{1,\gamma}$ surface in a neighborhood of each of its regular points. The constants C, γ depend on n, s , and β .

Theorem 2.5 was obtained by Caffarelli, Salsa and Silvestre in [CSS]. The main tool in the proof is to establish a version of Almgren's frequency formula for the "extension" of u to \mathbb{R}^{n+1} . However, Theorem 2.5 is proved in [CSS] in the case when $\varphi \in C^{2,1}$ (i.e. $\beta = 3$). When $s = 1/2$, Guillen proved Theorem 2.5 in [G]. In Section 6 we show that the Almgren's monotonicity formula still holds when $\beta > 1 + s$ and therefore sharpen the result in [CSS] and obtain Theorem 2.5.

Theorem 2.5 yields the following result for the two-membrane problem. When

$$K_1(y) = \frac{1}{|y|^{n+2s_1}}$$

we obtain the optimal regularity of the minimizing pair, i.e. $u_1 \in C^{1,s_1}$ and $u_2 \in C^{1+2s_2-s_1}$ and we can characterize the points on the free boundary

$$\Gamma := \partial\{u_1 = u_2\} \cap B_1,$$

as in the obstacle problem.

Theorem 2.6 (Optimal regularity). *Assume that the hypotheses of Theorem 2.3 hold and K_1 is as above. Then the conclusion of Theorem 2.3 holds with $\alpha_1 = 1 + s_1$. Moreover, the set of regular points of the free boundary Γ is locally a $C^{1,\gamma}$ surface.*

3. THE PROOF OF THEOREM 2.2

In this section we prove the Hölder regularity of the minimizing pair (u_1, u_2) . The parameters $\lambda, \Lambda, n, s_1, s_2$ are called universal and any constant depending only on these parameters is called universal as well and it is usually denoted by C, c (though it may change from line to line).

Proof of Theorem 2.2. The proof follows from the standard De Giorgi iteration technique. For simplicity we sketch it for $f_i = 0$ and $s_i < 1$, since the arguments carry on without difficulty to the case of nonzero f_i 's and when one or both operators are local.

Assume that

$$s_2 \geq s_1.$$

Step 1. Caccioppoli inequality. Let φ be a cutoff function supported in B_1 . The key observation is that for $\epsilon < 1$,

$$(u_1 + \epsilon\varphi^2 u_1^-, u_2 + \epsilon\varphi^2 u_2^-) \in \mathcal{A}.$$

Using the minimality of the pair (u_1, u_2) , we let $\epsilon \rightarrow 0$ and obtain

$$(3.1) \quad \mathcal{E}_{k_1}(u_1, \varphi^2 u_1^-) + \mathcal{E}_{k_2}(u_2, \varphi^2 u_2^-) \geq 0.$$

Notice that

$$-\mathcal{E}_k(u, \varphi^2 u^-) = \mathcal{E}_k(u^-, \varphi^2 u^-) + F_k(u),$$

and

$$F_k(u) := -\mathcal{E}_k(u^+, \varphi^2 u^-) = 2 \int \int \varphi^2(x) u^+(x) u^-(y) k(x, y) dx dy \geq 0.$$

We use the identity

$$(a - b)(p^2 a - q^2 b) = (ap - bq)^2 - ab(p - q)^2,$$

thus

$$\mathcal{E}_k(u^-, \varphi^2 u^-) = \mathcal{E}_k(\varphi u^-, \varphi u^-) - I_k(u)$$

with

$$I_k(u) = \int \int u^-(x) u^-(y) (\varphi(x) - \varphi(y))^2 k(x, y) dx dy \geq 0.$$

The identities above give

$$\mathcal{E}_k(\varphi u^-, \varphi u^-) + F_k(u) = -\mathcal{E}_k(u, \varphi^2 u^-) + I_k(u).$$

Next we bound above $I_k(u)$.

Assume that φ is the usual cutoff function with $\varphi = 1$ in B_r and $\varphi = 0$ outside $B_{r+\delta/2}$ for some $r \in (0, 1 - \delta]$. When both x and y are in $B_{r+\delta}$ we use that

$$u^-(x) u^-(y) (\varphi(x) - \varphi(y))^2 \leq C \delta^{-2} [(u^-(x))^2 + (u^-(y))^2] |x - y|^2.$$

When $x \in B_{r+\delta/2}$ and y lies outside $B_{r+\delta}$ (and symmetrically the other case), we use that

$$k(x, y) \leq C \delta^{-n-2s} \omega(y).$$

Thus, we see that $I_k(u)$ is bounded above by

$$I_k(u) \leq C \delta^{-2} \int_{B_{r+\delta}} (u^-)^2 dx + C \|u^-\|_{L^2(d\omega)} \delta^{-n-2s} \int_{B_{r+\delta}} u^- dx.$$

In this last inequality we used that $\|u^-\|_{L^1(d\omega)} \leq C \|u^-\|_{L^2(d\omega)}$. We use these relations for u_1 and u_2 in the energy inequality (3.1) together with the fact that $u_2^- \geq u_1^-$ in B_1 . We obtain the desired Caccioppoli inequality for u_2^- :

$$(3.2) \quad \mathcal{E}_{k_2}(\varphi u_2^-, \varphi u_2^-) + F_{k_2}(u_2) \leq C_0 \delta^{-n-2} \int_{B_{r+\delta}} [(u_2^-)^2 + M_0 u_2^-] dx.$$

with

$$M_0 := \|u_1^-\|_{L^2(d\omega_1)} + \|u_2^-\|_{L^2(d\omega_2)},$$

and C_0 universal. More generally if $v_m = u_2 + m$, we have

$$(3.3) \quad \mathcal{E}_{k_2}(\varphi v_m^-, \varphi v_m^-) + F_{k_2}(v_m) \leq C_0 \delta^{-n-2} \int_{B_{r+\delta}} [(v_m^-)^2 + A_m v_m^-] dx,$$

and

$$M_m = \|(u_1 + m)^-\|_{L^2(d\omega_1)} + \|(u_2 + m)^-\|_{L^2(d\omega_2)}.$$

Moreover, for all constants $m \geq 0$, $M_m \leq M_0$ hence

$$(3.4) \quad \mathcal{E}_{k_2}(\varphi v_m^-, \varphi v_m^-) + F_{k_2}(v_m) \leq C_0 \delta^{-n-2} \int_{B_{r+\delta}} [(v_m^-)^2 + M_0 v_m^-] dx.$$

Remark 3.1. Since u_2 is a subsolution for the \mathcal{L}_{k_2} operator, $v_m^+ := (u_2 - m)^+$ satisfies the same inequality (3.3) with the constant M_m replaced by $\|(u_2 - m)^+\|_{L^2(d\omega_2)}$.

Step 2. The first De Giorgi lemma. We write the first De Giorgi type lemma and provide a sketch of the proof (see also Lemma 3.1 in [CCV].)

Lemma 3.2 (L^∞ bound). *Assume $v_m := u_2 + m$ satisfies (3.4) for all $0 \leq m \leq 1$ and some $M_0 > 0$. There exists ϵ_0 depending on the universal parameters and M_0 such that if*

$$\|u_2^-\|_{L^2(B_1)} \leq \epsilon_0(M_0),$$

then

$$u_2^- \leq 1 \quad \text{in } B_{1/2}.$$

Proof. We apply (3.4), with $(j \geq 2)$

$$m = m_j := 1 - 2^{-j}, \quad r = r_j := \frac{1}{2} + 2^{-j}, \quad \delta = \delta_j := 2^{-j}.$$

Using that $F_{k_2}(v_{m_j}) \geq 0$ together with Sobolev inequality we get $(1/2^* = 1/2 - s_2/n)$

$$(3.5) \quad \left(\int_{B_{r_j}} (v_{m_j}^-)^{2^*} \right)^{2/2^*} \leq C_0 \delta_j^{-n-2} \int_{B_{r_j+\delta_j}} [(v_{m_j}^-)^2 + M_0 v_{m_j}^-] dx \\ := R_j.$$

Call,

$$a_j := \int_{B_{r_j}} (v_{m_j}^-)^2$$

and

$$A_j := \{v_{m_j} < 0\} \cap B_{r_j}.$$

Applying Holder's inequality to the left-hand-side of (3.5) and using the notation above we get

$$(3.6) \quad a_j \leq |A_j|^{\frac{2s_2}{n}} R_j \leq |A_j|^{\frac{2s_2}{n}} (C_0 2^{Mj} a_{j-1} + M_0 a_{j-1}^{1/2} |A_j|^{1/2}),$$

for some large M . Since on A_j , $v_{m_{j-1}} < -2^j$, we easily obtain that

$$a_{j-1} \geq |A_j| 2^{-2j}.$$

Thus, (3.6) gives (for some positive σ and with \bar{C} depending on the universal constants and M_0)

$$a_j \leq \bar{C} 2^{Mj} a_{j-1}^{1+\sigma}.$$

Standard De Giorgi iteration gives that if a_2 is small enough (depending on \bar{C}) $a_j \rightarrow 0$ as $j \rightarrow \infty$ and from this we deduce our claim. \square

Our minimization problem remains invariant after multiplication with a constant. Thus, after multiplication with a small constant we may apply Lemma 3.2 and obtain the L^∞ bound for u_2 in $B_{1/2}$.

Step 2. The second De Giorgi lemma and the Hölder continuity of u_2 . In order to obtain the Hölder continuity of u_2 , we need to iterate the next Lemma 3.3, and this is point where we need $s_2 \geq s_1$.

Notice that in general the minimization problem is not invariant after a dilation in the space variable. Indeed, if $\tilde{u}_i(x) = u_i(\rho x)$ then

$$(\tilde{u}_1, \tilde{u}_2) \quad \text{minimizes the energy} \quad \rho^{2(s_2-s_1)} \mathcal{E}_{k_1}(\tilde{u}_1, \tilde{u}_1) + \mathcal{E}_{k_2}(\tilde{u}_2, \tilde{u}_2).$$

Thus if $\rho \leq 1$ the arguments above apply and the Caccioppoli inequality (3.2) holds for \tilde{u}_2 with

$$\tilde{M} = \rho^{2(s_2-s_1)} \|\tilde{u}_1^-\|_{L^2(d\omega_1)} + \|\tilde{u}_2^-\|_{L^2(d\omega_2)} \leq \tilde{M}_0 := \|\tilde{u}_1^-\|_{L^2(d\omega_1)} + \|\tilde{u}_2^-\|_{L^2(d\omega_2)}.$$

Notice also that

$$(3.7) \quad \|\tilde{u}_i\|_{L^2(\mathbb{R}^n \setminus B_{1/\rho}, d\omega_i)} \sim \rho^{s_i} \|u_i\|_{L^2(\mathbb{R}^n \setminus B_1, d\omega_i)}.$$

Lemma 3.3 (Oscillation decay). *Assume that u_2 satisfies (3.3), for all constants m . Suppose that for some $R \geq 1$,*

$$u_1 \geq u_2 \quad \text{in } B_R,$$

and

$$|u_2| \leq 1 \quad \text{in } B_R, \quad \|u_1^-\|_{L^2(\mathbb{R}^n \setminus B_R, d\omega_1)} \leq \mu, \quad \|u_2^-\|_{L^2(\mathbb{R}^n \setminus B_R, d\omega_2)} \leq \mu,$$

with μ universal. Then in B_1 either $u_2 \leq 1 - \mu$ or $u_2 \geq -1 + \mu$.

Proof. Let us assume first that

$$|u_2| \leq 1 \quad \text{in } \mathbb{R}^n.$$

Assume that

$$(3.8) \quad |\{u_2 > 0\} \cap B_1| \geq \frac{1}{2}|B_1|.$$

We will show that there is a universal constant η such that $u_2 \geq -1 + \eta$ in B_1 . Let,

$$v_j := 2^j(u_2 + (1 - 2^{-j})), \quad A_j := \{v_j < 0\} \cap B_1.$$

We aim to show that there is a large enough j such that

$$(3.9) \quad |A_{j+1}| \leq \delta_0$$

with δ_0 universal to be made precise later.

Assume by contradiction that

$$|A_{j+1}| > \delta_0$$

and let us choose $\delta \ll \delta_0$ so that

$$(3.10) \quad |A_{j+1} \cap B_{1-\delta}| \geq \frac{\delta_0}{2}.$$

By Caccioppoli inequality (3.2) for v_j we obtain

$$(3.11) \quad F_{k_2}(v_j) \leq C\delta^{-n-2}$$

where we have used that $v_j^- \leq 1$ in \mathbb{R}^n , and that $u_1 \geq u_2$, so that the corresponding constant M_j in (3.2) is bounded by a universal constant \bar{M} .

On the other hand,

$$\begin{aligned} F_{k_2}(v_j) &:= 2 \int \int \varphi^2(x) v_j^+(x) v_j^-(y) k_2(x, y) dx dy \geq \\ &c \int_{B_1} v_j^+(y) dy \int_{A_{j+1} \cap B_{1-\delta}} v_j^-(x) dx \geq \\ &c(2^j - 1)|A_{j+1} \cap B_{1-\delta}||B_1| \geq 2^j c \delta_0. \end{aligned}$$

In the third inequality above we used that

$$v_j^- \geq \frac{1}{2} \quad \text{on } A_{j+1}$$

and (3.8).

Thus, (3.10) is violated if j is large enough. Denote such j by \bar{j} .

Now we can apply Lemma 3.2 to $v_{\bar{j}+1}$ and choose $\delta_0 = \epsilon_0(2\bar{M})$ where \bar{M} is the universal constant that bounds all the M_j 's (as observed above.) We obtain the conclusion with $\eta = 2^{-(\bar{j}+1)}$.

Now assume that $|u_2| \leq 1$ in B_R and $u_1 \geq u_2$ in B_R , for $R \geq 1$. Let also

$$\|u_i^-\|_{L^2(\mathbb{R}^n \setminus B_R, d\omega_i)} \leq \epsilon.$$

Then, for ϵ small enough the argument above still holds for the fixed \bar{j} . Indeed one can still guarantee that $M_{\bar{j}} \leq 2\bar{M}$ for ϵ small enough.

Finally, if (3.8) does not hold, then we can work with the Caccioppoli inequality for $(u_2 - m)^+$ and obtain that u_2 separates from the top (see Remark 3.1). \square

Finally we can iterate Lemma 3.3 and obtain the interior C^α Hölder continuity of u_2 . Indeed, after a multiplication by a constant we may assume that $\|u_i\|_{L^2(d\omega_i)}$ are sufficiently small and $|u_2| \leq 1$ in $B_{1/2}$. Then we perform an initial dilation of size R_0 , and we may apply Lemma 3.3. Notice that the hypotheses are satisfied thanks to (3.7). Moreover it is easy to check that the hypotheses hold for the sequence of Hölder rescalings

$$\left(\frac{2}{2-\mu}\right)^{m-1} u_2(R_0^{-m}x) + \text{const}, \quad m = 1, 2, \dots$$

provided that R_0 is chosen sufficiently large, and we may apply Lemma 3.3 indefinitely.

Step 3. The second De Giorgi lemma and the Hölder continuity of u_1 . Next we obtain the Hölder continuity of u_1 by thinking that $u_2 \in C^\alpha$ is a fixed obstacle lying above, and u_1 minimizes $\mathcal{E}_{k_1}(u_1, u_1)$ among admissible functions.

Notice that since $|u_2| \leq 1$ and $u_1 \geq u_2$ we can obtain an L^∞ bound for u_1 by applying the (standard) first De Giorgi lemma to $(u_1 - 1)^+$. Indeed in the set $u_1 > 1$, u_1 solves the equation $\mathcal{L}_{k_1} u_1 = 0$.

The Hölder continuity of u_1 follows by iterating the following version of the oscillation decay lemma.

Lemma 3.4. *Assume that for some $R \geq 1$*

$$|u_1| \leq 1 \quad \text{in } B_R, \quad \|u_1\|_{L^2(\mathbb{R}^n \setminus B_R, d\omega)} \leq \mu, \quad \text{osc}_{B_1} u_2 \leq 1/4.$$

Then in B_1 either $u_1 \leq 1 - \mu$ or $u_1 \geq -1 + \mu$.

The proof of Lemma 3.4 is a variation of the proof above. Indeed, if $u_2(0) \geq -\frac{1}{2}$ then the conclusion is obvious since $u_1 \geq u_2 \geq -\frac{3}{4}$.

If $u_2(0) \leq -\frac{1}{2}$, we distinguish two cases. When $|\{u_1 > 0\} \cap B_1| > 1/2$, we use that $\mathcal{L}_{k_1} u_1 \leq 0$ hence we apply De Giorgi technique to conclude that $u_1 \geq -1 + \mu$.

Otherwise, since $u_2 \leq -\frac{1}{4}$ in B_1 , u_1 is not constrained in the set $\{u_1 > 0\}$ and $\mathcal{L}_{k_1} u_1 = 0$ there. Again, we can apply De Giorgi technique and conclude $u_1 \leq 1 - \mu$. \square

4. TRANSLATION INVARIANT KERNELS AND VISCOSITY SOLUTIONS

In this section we investigate further properties of the minimizing pair (u_1, u_2) when the kernels k_i are more regular. More precisely, from now on we assume that the kernel k used in the definition of the energy \mathcal{E}_k in (2.2) is translation invariant i.e.

$$k(x, y) = K(x - y).$$

Here the kernel K satisfies $K(y) = K(-y)$ and it is comparable to the kernel of $(-\Delta)^s$ i.e.

$$(4.1) \quad \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \quad 0 < \lambda \leq \Lambda.$$

The integro-differential operator associated to this kernel can be written as

$$(4.2) \quad \mathcal{L}_K w(x) := PV \int_{\mathbb{R}^n} (w(y) - w(x)) K(y - x) dy.$$

Notice that the value $\mathcal{L}_K w(x)$ is well-defined as long as $w \in L^1(\mathbb{R}^n, d\omega)$ and w is $C^{1,1}$ at x .

In the case $s = 1$, of local operators defined in (2.3), we assume that the matrix A is constant, and therefore \mathcal{L}_A is a second order operator with constant coefficients.

4.1. Viscosity properties of the minimizing pair. To study further regularity of the minimizing pair, we adopt the point of view of viscosity solutions.

Definition 4.1. Given a function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, upper (lower) semicontinuous in \bar{B}_1 and a C^2 function ϕ defined in a neighborhood N of a point $x \in B_1$, we say that ϕ touches w by above (resp. below) at x if

$$\phi(x) = w(x), \quad \phi(y) > w(y) \quad (\phi(y) < w(y)) \quad \text{for every } y \in N \setminus \{x\}.$$

We remark that at any point x where w is touched by above or below, $\mathcal{L}_K w(x)$ is well-defined, though it may be infinite. Indeed, say w is touched by below by ϕ at x then

$$\mathcal{L}_K w(x) = \int_0^\infty a_w(r) r^{-1-2s} dr \in (-\infty, +\infty]$$

where $a_w(r)$ represents the averages of w on ∂B_r

$$a_w(r) = \oint_{\partial B_r(x)} (w(y) - w(x)) K(y - x) r^{n+2s} dy$$

and for r small (since K is symmetric)

$$a_w(r) \geq a_\phi(r) \geq -Cr^2.$$

Definition 4.2. A function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, upper (lower) semicontinuous in \bar{B}_1 , is said to be a viscosity subsolution (supersolution) to $\mathcal{L}_K w = f$, f continuous in B_1 , and we write $\mathcal{L}_K w \geq f$ ($\mathcal{L}_K w \leq f$), if at any point $x \in B_1$ where w is touched by above (resp. below) by a quadratic polynomial P , we have

$$\mathcal{L}_K w(x) \geq f(x), \quad (\mathcal{L}_K w(x) \leq f(x)).$$

A viscosity solution is a function w that is both a subsolution and a supersolution.

Next we show that distributional supersolutions (subsolutions) are also viscosity supersolutions (subsolutions). We sketch the proof since we will use the same argument in a slightly different context.

Lemma 4.3. Assume that $\mathcal{L}_K w \leq f$ in the distribution sense with w, f continuous functions in B_1 . Then $\mathcal{L}_K w \leq f$ in the viscosity sense.

Proof. Assume for simplicity that $f = 0$. Let P be a quadratic polynomial touching w strictly by below at say 0. Let $P_\epsilon := P + \epsilon$ and denote by

$$w_\epsilon := \max\{w, P_\epsilon\}.$$

and,

$$\varphi_\epsilon := w_\epsilon - w \geq 0.$$

From the hypothesis $\mathcal{E}_K(\varphi_\epsilon, w) \geq 0$ thus

$$\mathcal{E}_K(\varphi_\epsilon, w_\epsilon) = \mathcal{E}_K(\varphi_\epsilon, w) + \mathcal{E}_K(\varphi_\epsilon, \varphi_\epsilon) \geq 0.$$

Since on the support of φ_ϵ we have that w_ϵ is $C^{1,1}$ by below, we can integrate by parts $\mathcal{E}_K(\varphi_\epsilon, w_\epsilon)$ and obtain

$$(4.3) \quad \int_{A_\epsilon} \varphi_\epsilon(x) \mathcal{L}_K w_\epsilon(x) dx \leq 0,$$

where $A_\epsilon := \{x : w < P_\epsilon\}$. Fix $\delta > 0$, thus $A_\epsilon \subset B_\delta$, for all ϵ small. We use that $w_\epsilon \geq P_\epsilon$ in B_δ , $w_\epsilon = w$ outside B_δ , hence for $x \in A_\epsilon$,

$$(4.4) \quad \begin{aligned} \mathcal{L}_K w_\epsilon(x) &\geq \int_{B_\delta} (P_\epsilon(y) - P_\epsilon(x)) K(y-x) dy + \int_{\mathbb{R}^n \setminus B_\delta} (w(y) - P_\epsilon(x)) K(y-x) dy \\ &\geq \int_{\mathbb{R}^n \setminus B_\delta} (w(y) - w(0)) K(y) dy + o_\epsilon(1) + O(\delta^{2-2s}), \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

with $o_\epsilon(1) \rightarrow 0$ as $\epsilon \rightarrow 0$. Combining this estimate with (4.3), and using that $\varphi_\epsilon \geq 0$, we obtain that

$$\mathcal{L}_K w(0) \leq 0,$$

after letting ϵ and then δ go to zero. \square

By Lemma 4.3, if (u_1, u_2) is a minimizing pair and f_i are continuous functions then (see (2.5)-(2.6))

$$(4.5) \quad \mathcal{L}_{K_1} u_1 \leq f_1, \quad \mathcal{L}_{K_2} u_2 \geq f_2, \quad \text{in } B_1,$$

$$\mathcal{L}_{K_1} u_1 = f_1, \quad \mathcal{L}_{K_2} u_2 = f_2, \quad \text{in the open set } \{u_1 > u_2\}.$$

in the viscosity sense. Next we prove a similar statement in the closed set

$$(4.6) \quad E := \{u_1 = u_2\}.$$

Lemma 4.4. *Assume that u_2 is touched by below at a point $x_0 \in \{u_1 = u_2\} \cap B_1$ by a C^2 function. Then*

$$\mathcal{L}_{K_1} u_1(x_0) + \mathcal{L}_{K_2} u_2(x_0) \leq f_1(x_0) + f_2(x_0).$$

We remark that, since $u_1 \geq u_2$, u_1 is touched by below at x_0 by the same C^2 function, thus $\mathcal{L}_{K_1} u_1(x_0)$ is well defined.

Proof. We argue as above. Assume for simplicity that $f_1 = 0$, $f_2 = 0$, $x_0 = 0$, and let P be a quadratic polynomial touching u_2 strictly by below at 0. Let $P_\epsilon := P + \epsilon$ and denote by

$$u_i^\epsilon := \max\{u_i, P_\epsilon\}, \quad \varphi_i^\epsilon := u_i^\epsilon - u_i.$$

By minimality,

$$\sum_i (\mathcal{E}_{K_i}(u_i^\epsilon, u_i^\epsilon) - \mathcal{E}_{K_i}(u_i, u_i)) \geq 0,$$

thus

$$\sum_i \mathcal{E}_{K_i}(\varphi_i^\epsilon, u_i^\epsilon) \geq \frac{1}{2} \sum_i \mathcal{E}_{K_i}(\varphi_i^\epsilon, \varphi_i^\epsilon) \geq 0$$

After integrating by parts the terms $\mathcal{E}_K(\varphi^\epsilon, u^\epsilon)$ we get,

$$\sum_i \int_{A_i^\epsilon} \varphi_i^\epsilon (\mathcal{L}_{K_i} u_i^\epsilon) dx \leq 0,$$

where $A_i^\epsilon := \{u_i < P_\epsilon\}$. Arguing as (4.4) in Lemma 4.3 we obtain that

$$\sum_i \left(\int_{A_i^\epsilon} \varphi_i^\epsilon \right) (\mathcal{L}_{K_i} u_i(0) + o_\delta(1) + o_\epsilon(1)) \leq 0,$$

with

$$o_\delta(1) \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \quad o_\epsilon(1) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Since we already know that $\mathcal{L}_{K_1} u_1(0) \leq 0$ and also $0 < \varphi_1^\epsilon \leq \varphi_2^\epsilon$, we get the desired inequality after dividing by $\int \varphi_2^\epsilon$ and then letting $\epsilon \rightarrow 0$, $\delta \rightarrow 0$. \square

4.2. Viscosity formulation of the two membranes problem. Next we show that we can formulate the two membranes problem in a non-variational setting. With this approach we may consider the two membranes problem for nonlinear operators Iu or $F(D^2u)$ (instead of \mathcal{L}_K) which do not have necessarily a variational structure.

Below we show that the following conditions in B_1

$$(4.7) \quad u_1 \geq u_2, \quad \mathcal{L}_{K_1} u_1 \leq f_1, \quad \mathcal{L}_{K_2} u_2 \geq f_2$$

$$(4.8) \quad \mathcal{L}_{K_i} u_i = f_i \quad \text{in } \{u_2 < u_1\}, \quad \mathcal{L}_{K_1} u_1 + \mathcal{L}_{K_2} u_2 = f_1 + f_2,$$

determine the pair (u_1, u_2) uniquely.

We always assume that outside B_1 , $u_i = u_i^0$ are prescribed with $u_i^0 \in L^1(\mathbb{R}^n, d\omega)$ and continuous near ∂B_1 , and $u_1^0 \geq u_2^0$ near ∂B_1 . Also we assume that f_i 's are continuous and bounded in B_1 .

Definition 4.5. We say that (w_1, w_2) is a viscosity subsolution to (4.7)-(4.8) if w_i are continuous in a neighborhood of \bar{B}_1 , and in B_1 we have $w_2 \leq w_1$, and

$$(4.9) \quad \mathcal{L}_{K_2} w_2 \geq f_2,$$

$$(4.10) \quad \mathcal{L}_{K_1} w_1 + \chi_E \mathcal{L}_{K_2} w_2 \geq f_1 + \chi_E f_2 \quad \text{with } E := \{w_1 = w_2\}.$$

Similarly, we define the notion of viscosity supersolution for the two membranes problem. Equation (4.10) is understood as a differential inequality for w_1 which depends on w_2 . Notice that at a point $x_0 \in E$ where w_1 has a tangent C^2 function ϕ by above, the same function is tangent also to w_2 at x_0 , and therefore (4.10) provides an integro-differential inequality involving ϕ at x_0 . Precisely we require that when we replace w_i by ϕ in any δ neighborhood of x_0 the inequality (4.10) is satisfied at x_0 .

In the next lemma we show that even though the inequality (4.10) contains the discontinuous term χ_E , the notion of subsolution is preserved under uniform limits.

Lemma 4.6. Assume that (w_1^k, w_2^k) is a sequence of subsolutions with right hand sides (f_1^k, f_2^k) . Assume that w_i^k, f_i^k converge uniformly on compact sets of B_1 to \bar{w}_i, \bar{f}_i and that $w_i^k \rightarrow \bar{w}_i$ weakly in $L^1(\mathbb{R}^n, d\omega)$. Then (\bar{w}_1, \bar{w}_2) is a subsolution.

Proof. Clearly \bar{w}_2 satisfies (4.9). Assume that $\phi \in C^2$ touches strictly by above \bar{w}_1 at some point \bar{x} . Denote $\bar{E} := \{\bar{w}_1 = \bar{w}_2\}$.

If $\bar{x} \notin \bar{E}$ then we obtain as usual

$$(4.11) \quad \mathcal{L}_{K_1} \bar{w}_1(\bar{x}) \geq \bar{f}_1(\bar{x}),$$

and we are done.

If $\bar{x} \in \bar{E}$ we need to show that

$$\sum_i \mathcal{L}_{K_i} \bar{w}_i(\bar{x}) \geq \sum_i f_i(\bar{x}).$$

We slide the graph of ϕ by above till it touches w_1^k at x_k , and then $x_k \rightarrow \bar{x}$. We distinguish two cases: either $x_k \in E_k$ or $x_k \notin E_k$ for infinitely many k 's. In the first case we obtain the inequality above by writing it for the w_i^k at x_k and letting $k \rightarrow \infty$. In the second case we obtain (4.11) which combined with (4.9) for \bar{w}_2 gives the desired inequality again. \square

Lemma 4.7. *Assume that (w_1^k, w_2^k) , $k = 1, 2$ are two pairs of subsolutions, and let $\bar{w}_i = \max_k w_i^k$, $\bar{f}_i = \min_k f_i^k$. Then (\bar{w}_1, \bar{w}_2) is a subsolution.*

Proof. Notice that $\bar{E} := \{\bar{w}_1 = \bar{w}_2\} \subset E_1 \cup E_2$, $E_k := \{w_1^k = w_2^k\}$, $k = 1, 2$, and then the rest of the proof it is straightforward to check. \square

In view of the lemma above we can use the standard method of sup-convolutions (see [CC], [CS1]) and approximate a subsolution (w_1, w_2) with right hand side (f_1, f_2) by a sequence of semiconvex subsolutions $(w_1^\epsilon, w_2^\epsilon)$ and right hand side $(f_1^\epsilon, f_2^\epsilon)$.

Precisely, $(w_1^\epsilon, w_2^\epsilon)$ satisfies:

- a) has the same boundary data outside B_1 as the original pair,
- b) is a subsolution in $B_{1-\epsilon}$ and each w_i^ϵ is uniformly $C^{1,1}$ by below.
- c) $w_i^\epsilon \rightarrow w_i$, $f_i^\epsilon \rightarrow f_i$ uniformly in \bar{B}_1 as $\epsilon \rightarrow 0$.

Next we prove the following comparison principle.

Lemma 4.8 (Maximum principle). *Assume that (w_1, w_2) is a subsolution and (v_1, v_2) is a supersolution to (4.7)-(4.8) and $w_i \leq v_i$ outside B_1 . Then $w_i \leq v_i$ also in B_1 .*

Proof. We translate down the graphs of the pair (w_1, w_2) in \bar{B}_1 and then we move them up till either w_1 touches v_1 or w_2 touches v_2 for the first time.

Assume by contradiction that the first contact point occurs in the interior of B_1 . After regularizing the functions w_i , v_i as above and relabeling the translates by w_1 , w_2 we may assume we are in the following situation:

$$w_i \leq v_i, \quad w_2(x_0) = v_2(x_0) \quad \text{for some } x_0 \in B_1,$$

(w_1, w_2) is a strict subsolution and (v_1, v_2) is a strict supersolution at x_0 , and w_i , v_i are $C^{1,1}$ at x_0 . If at least one of the operators is local then we may assume that all the functions are C^2 at x_0 after subtracting locally a small linear function from one of the pairs, see [CC]. Let $E_w := \{w_1 = w_2\}$, $E_v := \{v_1 = v_2\}$ and we distinguish 2 cases.

Case 1: $x_0 \notin E_v$. Then we contradict the inequalities for $\mathcal{L}_{K_2} w_2$ and $\mathcal{L}_{K_2} v_2$ at x_0 .

Case 2: $x_0 \in E_v$. Then

$$w_1(x_0) \leq v_1(x_0) = v_2(x_0) = w_2(x_0),$$

thus $x_0 \in E_w$ as well. Now we contradict the inequalities for the sum of the two operators at x_0 . \square

Proposition 4.9 (Existence and uniqueness of viscosity solutions). *Let $u_i^0 \in L^1(\mathbb{R}^n, d\omega_i)$ be continuous in a neighborhood of ∂B_1 , and let f_i be continuous and bounded in B_1 . Then there exists a unique viscosity solution pair (u_1, u_2) to the two membranes problem (4.7)-(4.8).*

Proof. The proof follows the standard Perron's method and we will not sketch the details. We only mention that the continuity of u_i^0 in a neighborhood of ∂B_1 allows us to construct continuous upper and lower barriers for the subsolutions and supersolutions (see [RS]). Using this we can replace each subsolution by a larger subsolution with a fixed modulus of continuity in \bar{B}_1 , and therefore the largest subsolution will have the same modulus of continuity. \square

4.3. The case of different order operators. Next we establish the $C^{2s_2-\epsilon}$ interior regularity of u_2 in the case when $s_2 > s_1$.

Let (u_1, u_2) be a viscosity solution in B_2 , and assume that

$$\|u_i\|_{L^1(d\omega_i)} \leq 1, \quad \|f_i\|_{L^\infty(B_2)} \leq 1.$$

Since u_2 is a subsolution, we use the weak Harnack inequality (see Lemma 5.2 below) and obtain that $u_2 \leq C$ in $B_{3/2}$. This means that u_1 is a subsolution in the set $\{u_1 > C\} \cap B_{3/2}$, hence we apply Lemma 5.2 one more time and bound u_1 by above in B_1 . Similarly we bound u_i by below and obtain

$$\|u_i\|_{L^\infty(B_1)} \leq C.$$

Let

$$(4.12) \quad v := \chi_{B_1} u_2$$

be the restriction of u_2 to B_1 , and $x \in E \cap B_{1/2}$ (see (4.6)). Then, since $v \leq u_1$ in B_1 , and $v(x) = u_1(x)$ we find

$$\mathcal{L}_{K_1} u_1(x) \geq \mathcal{L}_{K_1} v(x) + \int_{CB_1} (u_1(y) - v(x)) K(y-x) dy,$$

hence

$$\mathcal{L}_{K_1} u_1(x) \geq \mathcal{L}_{K_1} v(x) - C.$$

Moreover, for any $x \in B_{1/2}$ we have

$$|\mathcal{L}_{K_2} u_2(x) - \mathcal{L}_{K_2} v(x)| \leq C,$$

in the viscosity sense. Combining the last two inequalities with the fact that u_2 is a subsolution and (u_1, u_2) is a supersolution pair in the sense of Definition 4.5 we obtain the following corollary.

Corollary 4.10. *The function v defined in (4.12) satisfies in $B_{1/2}$*

$$(4.13) \quad \mathcal{L}_{K_2} v \geq -M,$$

$$(4.14) \quad \mathcal{L}_{K_2} v + \chi_E \mathcal{L}_{K_1} v \leq M$$

with M a constant depending on n, s_i, λ, Λ .

Inequality (4.14) contains the discontinuous term χ_E and it is understood in the viscosity sense. Precisely, if v admits a tangent C^2 function by below at a point x , then we satisfy two different inequalities depending whether or not x is in E .

Since $s_2 > s_1$ then the term $\chi_E \mathcal{L}_{K_1} v$ can be treated as a perturbation. Then (4.13)-(4.14) can be thought heuristically as saying that $\mathcal{L}_{K_2} v \in L^\infty$, and we can infer that $v \in C^\beta$ for any $\beta < 2s_2$. We use the convention that when $\beta \in (1, 2)$, the class C^β denotes the class $C^{1, \beta-1}$. We prove this statement rigorously in the next proposition.

Proposition 4.11. *Assume $s_2 > s_1$, and that v is a continuous function supported in B_1 which satisfies (4.13)-(4.14) for some closed set E . Then $v \in C^\beta$ for any $\beta < 2s_2$ and*

$$\|v\|_{C^\beta(B_{1/4})} \leq C(\|v\|_{L^\infty} + M),$$

with C a constant depending on n, s_i, λ, Λ and β .

Proof. The lemma can be deduced from the arguments of Caffarelli and Silvestre in [CS2]. Since their results do not apply directly to our setting, we will sketch the proof of the proposition for completeness.

After multiplication by a small constant we may assume that $M = 1$ and $\|v\|_{L^\infty(B_1)}$ is sufficiently small.

We need to show that if for all balls B_r with $r = 2^{-l}, l = 0, 1, \dots, k$ for some $k \geq k_0$, we have

$$(4.15) \quad |v - l_r| \leq r^\beta \quad \text{in } B_r,$$

with l_r a constant if $\beta < 1$ or a linear function if $\beta > 1$, and $l_1 \equiv 0$, then (4.15) holds also in $B_{\rho r}$ for some $l_{\rho r}$ where $\rho = 2^{-m_0}$. Here the constants m_0, k_0 depend on β and the universal constants. Then we can iterate (4.15) indefinitely and obtain the desired conclusion.

The existence of k_0 is obtained by compactness. Indeed, assume that (4.15) holds up to $r = r_k$ for some large k . Notice that the coefficients of l_r are bounded by a fixed constant, hence the rescaling

$$\tilde{v}(x) = r^{-\beta}(v - l_r)(rx),$$

satisfies

$$\|\tilde{v}\|_{L^\infty(B_1)} \leq 1, \quad |\tilde{v}(x)| \leq C_0 |x|^\beta \quad \text{outside } B_1.$$

Next we write (4.13)-(4.14) in terms of \tilde{v} . We have

$$\mathcal{L}_{K_2} v(x) = \mathcal{L}_{K_2} \left(l_r + r^\beta \tilde{v} \left(\frac{x}{r} \right) \right) = r^{\beta-2s_2} \mathcal{L}_{\tilde{K}_2} \tilde{v} \left(\frac{x}{r} \right).$$

We estimate $\mathcal{L}_{K_1} v$ by writing

$$v(x) = \chi_{B_1} v(x) = \chi_{B_1} l_r(x) + \chi_{B_1 \setminus B_{2r}} r^\beta \tilde{v} \left(\frac{x}{r} \right) + \chi_{B_{2r}} r^\beta \tilde{v} \left(\frac{x}{r} \right) =: v_1 + v_2 + v_3.$$

We have $|\mathcal{L}_{K_1} v_1| \leq C$ in B_r . Without loss of generality we may assume that $\beta > 2s_1$ which, by the growth of \tilde{v} outside B_1 gives $|\mathcal{L}_{K_1} v_2| \leq C$ in B_r . Also

$$\mathcal{L}_{K_1} v_3(x) = r^{\beta-2s_1} \mathcal{L}_{\tilde{K}_1} (\chi_{B_2} \tilde{v}) \left(\frac{x}{r} \right).$$

In conclusion \tilde{v} satisfies in B_1 the following inequalities

$$(4.16) \quad \mathcal{L}_{\tilde{K}_2} \tilde{v} \geq -C r^{2s_2-\beta},$$

$$(4.17) \quad \mathcal{L}_{\tilde{K}_2} \tilde{v} + r^{2(s_2-s_1)} \chi_E \mathcal{L}_{\tilde{K}_1} (\chi_{B_2} \tilde{v}) \leq C r^{2s_2-\beta}.$$

The function \tilde{v} is both a subsolution and a supersolution for integro-differential equations with measurable kernels and bounded right hand side. Since $r^{2(s_2-s_1)}$ is small, the two operators above are bounded by two extremal Pucci operators of order $2s_2$. We apply the Harnack inequality for integro-differential equations from [CS1] and obtain that \tilde{v} is uniformly Hölder continuous in $B_{3/4}$. This means that as $r \rightarrow 0$ (or equivalently as $k \rightarrow \infty$), the corresponding \tilde{v} 's converge uniformly on a subsequence to a limit \bar{v} . We claim that \bar{v} satisfies

$$|\mathcal{L}_{\bar{K}}(\chi_{B_{3/4}}\bar{v})| \leq C \quad \text{in } B_{1/2},$$

where \bar{K} is the weak limit of the \tilde{K}_2 's.

Indeed, let $\tilde{w} := \chi_{B_{3/4}}\tilde{v}$, then (4.16)-(4.17) give

$$\mathcal{L}_{\tilde{K}_2}\tilde{w} \geq -C, \quad \mathcal{L}_{\tilde{K}_2}\tilde{w} + r^{2(s_2-s_1)}\chi_{\tilde{E}}\mathcal{L}_{\tilde{K}_1}\tilde{w} \leq C,$$

with $r^{2(s_2-s_1)} \rightarrow 0$. Now we can pass to the limit in these inequalities and use that $\mathcal{L}_{\tilde{K}_2}\psi(x) \rightarrow \mathcal{L}_{\bar{K}}\psi(x)$ for any test function $\psi \in C^2$ near x , and obtain the claim.

The existence of $l_{\rho r}$ with $\rho = 2^{-m_0}$ universal, now follows from the $C^{\beta+\epsilon}$ estimates, with $\beta + \epsilon < 2s_2$, of the solution \bar{v} above, see Proposition 7.1, part a). \square

Remark 4.12. We are not concerned in obtaining estimates that remain uniform as the order of the operators approaches 2.

The Harnack inequality for \tilde{v} can be checked also directly by using the methods of Silvestre in [S2]. For this we slide parabolas by above and below till they touch the graph of \tilde{v} . Then we use the equation only at these points and show that the oscillation of \tilde{v} decays at a geometric rate as we restrict to dyadic balls. We will use this method more precisely in Section 5, see Step 1 in Proposition 5.6.

We remark the same argument works as well in the case when \mathcal{L}_{K_2} is a local operator, and then we need to use the ABP measure estimate, see [Sa] for example.

Proposition 4.11 provides the initial $C^{2s_2-\epsilon}$ interior regularity of the function u_2 . Now we can view the function u_1 as the solution to the obstacle problem with obstacle u_2 . Therefore in our analysis it is important to obtain regularity of solutions to the obstacle problem with not necessarily C^2 obstacle. In the next two sections we show that u_1 is as regular as the obstacle up to $C^{\max\{1, 2s_1\}+\epsilon}$ regularity in the case of translation invariant kernels, and up to C^{1+s_1} -regularity in the case of the fractional Laplacian.

Then we can successively improve the regularity of u_2 and u_1 and obtain Theorems 2.3 and 2.6.

Proof of Theorem 2.3. From Theorem 5.1 in Section 5 we have that u_1 is as regular as u_2 up to $C^{\max\{1, 2s_1\}+\epsilon}$ regularity, and $u_2 \in C^{2s_2-\epsilon}$ by Proposition 4.11. From the Schauder estimates for the equation $\mathcal{L}_K u = f$, see Proposition 7.1 in the Appendix, this implies that $\mathcal{L}_{K_1} u_1 \in C^\epsilon$. Thus $\mathcal{L}_{K_2} u_2 \in C^\epsilon$ which gives $u_2 \in C^{2s_2+\epsilon}$. Now we can iterate this argument and obtain the desired conclusion. \square

5. THE OBSTACLE PROBLEM FOR TRANSLATION INVARIANT KERNELS

In this section we make a detour to provide two regularity results for the general obstacle problem in the case of symmetric, translation invariant operators \mathcal{L}_K as above. We then apply these results to the two membranes problem.

In addition to (4.1) we need to impose the extra regularity assumption on K , i.e.

$$(5.1) \quad |\nabla K(y)| \leq \Lambda |y|^{-(n+1+2s)}.$$

Assume that u is a solution of the obstacle problem in B_1 with obstacle φ by below. Precisely we assume that u, φ are continuous in B_1 , $u \in L^1(\mathbb{R}^n, d\omega)$, and

$$(5.2) \quad u \geq \varphi \quad \text{in } B_1,$$

$$(5.3) \quad \mathcal{L}_K u \leq f \quad \text{in } B_1, \quad \text{and} \quad \mathcal{L}_K u = f \quad \text{in } \{u > \varphi\} \cap B_1.$$

Our main result of this section says that up to C^{1,ϵ_0} with ϵ_0 universal, the solution u is as regular as the obstacle φ . Moreover, in the case $s > \frac{1}{2}$, the C^{1,ϵ_0} regularity can be improved to $C^{2s+\epsilon_0}$.

Theorem 5.1. *Let u is a solution to the obstacle problem (5.2), (5.3), with kernel K that satisfies (4.1), (5.1), and assume that*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{0,1}(B_1)} \leq 1.$$

for some $\beta \neq 2s$.

Then $u \in C^\alpha(B_1)$ for $\alpha = \min\{\beta, \max\{1, 2s\} + \epsilon_0\}$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C,$$

where ϵ_0 depends on n, λ, Λ, s , and the constant C may depend also on β .

Before we proceed with the proof of Theorem 5.1 we write two versions of Harnack inequality for nonlocal equations which deal with L^∞ bounds for subsolutions.

Lemma 5.2. *Assume that v is continuous in $\overline{B_1}$, $\|v^+\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$, and*

$$\mathcal{L}_K v \geq -1 \quad \text{in } \{v > 1\} \cap B_1.$$

Then $v \leq C$ in $B_{1/2}$ with C depending only on n, s, λ, Λ .

Proof. After multiplication with a small constant we may replace 1 by δ_0 in the hypotheses above. We show that $v \leq \psi$ with

$$\psi(x) := (1 - |x|^2)^{-n}.$$

Assume by contradiction that when we slide the graph of ψ by above we touch the graph of v at some point $(x_0, v(x_0))$ above the original graph of ψ , i.e. there exists $t > 0$ such that $v \leq \psi_t$ in B_1 and $v(x_0) = \psi_t(x_0)$ for some x_0 , where $\psi_t := \psi + t$. Denote by

$$d := 1 - |x_0|,$$

and by l the tangent plane of ψ_t at x_0 . Then for $r \leq d/2$ we have

$$\begin{aligned} \int_{B_r(x_0)} (v(x) - v(x_0)) K(x - x_0) dx &\leq \int_{B_r(x_0)} (\Lambda(v - l)^+ - \lambda(v - l)^-) |x - x_0|^{-n-2s} dx \\ &\leq C d^{-n-2} r^{2-2s} - \lambda r^{-n-2s} \int_{B_r(x_0)} (v - l)^- dx. \end{aligned}$$

We use

$$\begin{aligned} \int_{B_r(x_0)} (v - l)^- dx &\geq \int_{B_r(x_0)} (l - v) dx \geq \psi_t(x_0) |B_r| - \int_{B_1} v^+ dx \\ &\geq C d^{-n} r^n - \delta_0, \end{aligned}$$

which, by taking $r = dc$ with c small, and $\delta_0 \ll c$ sufficiently small, we obtain

$$\int_{B_r(x_0)} (v(x) - v(x_0))K(x - x_0)dx \leq -cr^{-n-2s}.$$

On the other hand

$$\int_{\mathcal{C}B_r(x_0)} (v(x) - v(x_0))K(x - x_0)dx \leq \Lambda \int_{\mathcal{C}B_r(x_0)} v^+(x)|x - x_0|^{-n-2s}dx \leq C\delta_0 r^{-n-2s}.$$

From the last two inequalities we find

$$\mathcal{L}_K v(x_0) \leq -c,$$

and we reached a contradiction, provided that δ_0 is chosen sufficiently small. \square

We remark that in the proof we did not use the translation invariant properties of K , and clearly the proof holds for truncated kernels $\chi_{B_2}K$ as well. Also the assumption on the bound for the L^1 norm of v^+ in \mathbb{R}^n can be weakened to an L^1 bound for v^+ only on $\mathcal{C}B_{3/4}$. This can be seen by appropriately modifying the comparison function ψ in the proof.

We provide a version of Harnack inequality that follows from Lemma 5.2.

Lemma 5.3. *Assume that $v \geq 0$ in B_1 , $v(0) \leq 1$,*

$$\mathcal{L}_K v \leq \sigma \quad \text{in } B_1, \quad \mathcal{L}_K v \geq \sigma - 1 \quad \text{in } \{v > 1\} \cap B_1,$$

for some σ , and

$$\int |v|(\max\{1, |x|\})^{-(n+1+2s)} dx \leq 1.$$

Then $v \leq C$ in $B_{1/2}$ with C independent of σ .

Proof. Let $K_T = \chi_{B_2}K$ be the truncation of K , and we show that v and K_T satisfy the hypotheses of Lemma 5.2. We slide the parabola $x_{n+1} = -4|x|^2$ by below till it touches the graph of v at some point y_0 , and from our hypotheses above it follows that $y_0 \in B_{1/2}$, $v(y_0) \leq 1$, and

$$\mathcal{L}_{K_T} v(y_0) \geq -C.$$

For $y \in B_1$ we have

$$\mathcal{L}_K v(y) - \mathcal{L}_K v(y_0) \leq \mathcal{L}_{K_T} v(y) - \mathcal{L}_{K_T} v(y_0) + \int_{\mathcal{C}B_2} v(x)(K(x-y) - K(x-y_0))dx + C,$$

and from (5.1) we have that

$$|K(x-y) - K(x-y_0)| \leq C|x|^{-(n+1+2s)} \quad \text{if } x \in \mathcal{C}B_2.$$

Thus

$$\mathcal{L}_{K_T} v(y) \geq -C \quad \text{in } \{v > 1\} \cap B_1,$$

and the conclusion follows from Lemma 5.2. \square

Remark 5.4. We remark that if we slide a parabola $4C|x|^2$ by above and it touches the graph of v at some point y_1 for which $\mathcal{L}_K v(y_1) \geq \sigma - 1$ then by repeating the argument “upside-down” (i.e. for $-v$) we obtain $\mathcal{L}_{K_T} v(y) \leq C$ in B_1 .

We are now ready to prove Theorem 5.1, which is a direct consequence of Propositions 5.6 and 5.7 below. First we state the necessary Schauder estimates, which will be proved in the appendix.

Proposition 5.5 (Schauder estimates). *Let K be a symmetric kernel that satisfies (4.1), and assume that $v \in L^1(\mathbb{R}^n, d\omega)$ satisfies*

$$\mathcal{L}_K v = f \quad \text{in } B_1, \quad \|v\|_{L^\infty(B_1)} \leq 1.$$

a) *If $\|f\|_{L^\infty(B_1)} \leq 1$, $\|v\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$ then*

$$\|v\|_{C^\alpha(B_{1/2})} \leq C(\alpha), \quad \text{for any } \alpha < 2s.$$

b) *Assume that K satisfies (5.1). If*

$$\int_{CB_1} v |x|^{-(n+2s+1)} dx \leq 1, \quad [f]_{C^\gamma(B_1)} \leq 1, \quad \text{for some } \gamma \in (0, 1)$$

then

$$\|v\|_{C^\beta(B_{1/2})} \leq C(\gamma), \quad \text{with } \beta = 2s + \gamma,$$

provided that $2s + \gamma$ is not an integer.

c) *Conversely, if $\|v\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$, $\|v\|_{C^\beta(B_1)} \leq 1$ with β as above, then*

$$\|f\|_{C^\gamma(B_{1/2})} \leq C.$$

Proposition 5.5 can be easily deduced from the results of Serra in [Ser] where he obtained Schauder estimates for concave integro-differential equations with rough kernels (see also [K],[CS2]). We will sketch the proof in the Appendix, since its statement is slightly different than it usually appears in the literature and our setting is simpler than in [Ser].

Next, we prove the statement in Theorem 5.1, valid for all $s \in (0, 1)$, that is the following proposition.

Proposition 5.6. *Let u satisfy (5.2), (5.3) and assume that*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{0,1}(B_1)} \leq 1.$$

Then $u \in C^\alpha(B_1)$ for $\alpha = \min\{\beta, 1 + \epsilon_0\}$ and $\|u\|_{C^\alpha(B_{1/2})} \leq C$.

Proof. We sketch the proof below. In view of Lemma 5.2, we can assume without loss of generality that $\|u\|_{L^\infty} \leq 1$ in B_1 . In fact, after multiplication with a small constant, we may assume that all the norms in our assumptions and $\|u\|_{L^\infty}$ are bounded by δ_0 , sufficiently small to be made precise later.

Step 1: We show that $u \in C^{\alpha_0}$ for a small $\alpha_0 > 0$, by checking that the usual proof for Hölder continuity of solutions to nonlocal equations [S2] still applies in our case. Let us assume for simplicity that $0 \in \{u = \varphi\}$, $u(0) = 0$ and suppose that

$$(5.4) \quad u \leq r^{\alpha_0} = (1 - \delta)^l \quad \text{in } B_r, \quad \text{with } r = 2^{-l}, \quad \text{for all } l \leq k,$$

for some $k \geq k_0$. Then we need to show that (5.4) holds for $l = k + 1$ as well.

Indeed, the rescaling $\tilde{u}(x) := r^{-\alpha_0} u(rx)$ with $r = 2^{-k}$ satisfies in B_1 ($\alpha_0 \leq \beta$)

$$-\delta_0 \leq \tilde{u} \leq 1, \quad \mathcal{L}_{\tilde{K}} \tilde{u} \leq \delta_0, \quad \mathcal{L}_{\tilde{K}} \tilde{u} \geq -\delta_0 \quad \text{in } \{\tilde{u} > \delta_0\}.$$

Moreover,

$$(5.5) \quad \tilde{u} \leq (1 - \delta)^j, \quad \text{in } B_{2^j}, \quad j = 1, \dots, k,$$

$$(5.6) \quad \int_{\mathbb{R}^n \setminus B_{2^k}} \tilde{u} d\omega \leq (2^{-k})^{2s - \alpha_0} \delta_0.$$

In order to obtain the diminish of oscillation of \tilde{u} we compute $\mathcal{L}_K \tilde{u}$ at the two contact points x_0^- , x_0^+ obtained by sliding two paraboloids of opening 2δ by below and above till they touch the graph of \tilde{u} .

Precisely, we slide $P_t := 2\delta|x|^2 + t$, $t \leq 1$, from above. If no contact point occurs till $t = 1 - \frac{3}{2}\delta$, then

$$\tilde{u} \leq 1 - \delta \quad \text{in } B_{1/2}$$

and we obtain the desired diminish in oscillation. Let us consider then the case when the contact point x_0^+ occurs for $t > 1 - 3/2\delta$, that is near the top $x_{n+1} = 1$. Hence (say $\delta_0 < 1/4, \delta < 1/2$)

$$u(x_0^+) > \delta_0 \quad \text{and} \quad \mathcal{L}_{\tilde{K}}\tilde{u}(x_0^+) \geq -\delta_0.$$

Assume that

$$(5.7) \quad |\{\tilde{u} > \frac{1}{2}\} \cap B_1| < \frac{1}{2}|B_1|.$$

We show that

$$(5.8) \quad \mathcal{L}_{\tilde{K}}\tilde{u}(x_0^+) \leq -c$$

for c universal, provided that δ (hence α_0) is small enough. We thus reach a contradiction if δ_0 is small enough.

Indeed, for δ small,

$$\tilde{u} \leq P_t - \frac{1}{4}\chi_{\{\tilde{u} \leq \frac{1}{2}\}} \quad \text{in } B_1.$$

Hence,

$$\begin{aligned} \mathcal{L}_{\tilde{K}}\tilde{u}(x_0^+) &\leq \int_{B_1} (P_t(x) - P_t(x_0^+))\tilde{K}(x - x_0^+)dx - \frac{1}{4} \int_{\{\tilde{u} \leq \frac{1}{2}\} \cap B_1} \tilde{K}(x - x_0^+) \\ &\quad + \int_{\mathbb{R}^n \setminus B_1} (\tilde{u}(x) - \tilde{u}(x_0^+))\tilde{K}(x - x_0^+)dx := I_1 + I_2 + I_3. \end{aligned}$$

We first observe that $x_0^+ \in B_{3/4}$, since $\tilde{u} \leq 1$ and $t > 1 - 3/2\delta$.

It is easily seen that

$$I_1 \leq C_1\delta.$$

Moreover, from (5.7) we have

$$I_2 \leq -c_2.$$

Finally, we estimate I_3 as follows, and we recall that $k \geq k_0$ large.

$$I_3 \leq \sum_{j=1}^k \int_{B_{2^j} \setminus B_{2^{j-1}}} (\tilde{u}(x) - \tilde{u}(x_0^+))\tilde{K}(x - x_0^+)dx + \int_{\mathbb{R}^n \setminus B_{2^k}} \tilde{u}d\omega = I_3^1 + I_3^2.$$

To estimate I_3^1 we use (5.5) and get

$$I_3^1 \leq C \sum_{j=1}^k ((1 - \delta)^{-j} - 1 + \frac{3}{2}\delta)2^{-2sj} \leq c(\delta) \rightarrow 0, \quad \text{as } \delta \rightarrow 0.$$

Again, to estimate I_3^2 we use (5.5) and obtain

$$I_3^2 \leq (2^{-k})^{2s-\alpha_0} \rightarrow 0 \quad \text{for } k_0 \text{ large enough and } \delta \text{ (hence } \alpha_0) \text{ small.}$$

Combining the estimates above, we obtain the claim in (5.8) and reach a contradiction.

This implies that either the contact point does not occur near the top, and we are done, or (5.7) does not hold and

$$(5.9) \quad |\{\tilde{u} > \frac{1}{2}\} \cap B_1| \geq \frac{1}{2}|B_1|.$$

In this case, we slide $-2\delta|x^2|-t$ by below, $t \geq \delta_0$, and we work with the lower contact point x_0^- . Since $\tilde{u}(0) = 0$ we see that x_0^- occurs close to the bottom $x_{n+1} = -\delta_0$. With a similar computation as above, we obtain that

$$\mathcal{L}_{\tilde{K}}\tilde{u}(x_0^-) \geq c,$$

with c universal (δ chosen small). This contradicts that $\mathcal{L}_{\tilde{K}}\tilde{u}(x_0^-) \leq \delta_0$, if δ_0 is small. This means that (5.7) must hold and x_0^+ will occur far from the top, providing the diminish in the oscillation.

This establishes a uniform pointwise C^{α_0} -Holder continuity of u at all points on the contact set $\{u = \varphi\} \cap B_{1/2}$. It is easy to extend this modulus of continuity at all $x \in B_{1/4}$. We take the largest ball $B_\rho(x)$ included in $\{u > \varphi\}$ which is tangent to $\{u = \varphi\}$ at some point y , and then we apply the interior estimates in Proposition 5.5 to $\mathcal{L}_K u = f$ in $B_\rho(x)$ by using the modulus of continuity of u at y .

Step 2: We show that if $u \in C^\alpha$ for some $\alpha \leq 1$ then $u \in C^{\alpha+\epsilon_0}$ for some ϵ_0 universal, as long as $\alpha + \epsilon_0 \leq \beta$. Then we combine this claim and step 1, and obtain the desired conclusion.

The proof is similar to the one in Step 1, and uses the fact that the derivatives of u are “subsolutions”. Let us assume that the norms of the data are bounded by δ_0 and that

$$u(0) = \varphi(0) = 0, \quad \nabla\varphi(0) = 0 \quad \text{if } \beta > 1, \quad \text{and} \quad \|u\|_{C^\alpha(B_1)} \leq \delta_0.$$

We consider the difference quotients

$$u_h^e(x) := \frac{u(x + he) - u(x)}{h^\alpha},$$

where e is a unit vector and prove the following property.

Assume that for some $k \geq k_0$, we have for all $r = 2^{-l}$ with $l \leq k$

$$(5.10) \quad u_h^e \leq r^{\epsilon_0} = (1 - \delta)^l \quad \text{in } B_r, \text{ for all } h \leq r, |e| = 1.$$

Then (5.10) holds for $l = k + 1$ as well.

Fix $r = 2^{-k}$. The key observation is that

$$(5.11) \quad \mathcal{L}_K u_h^e \geq f_h^e \geq -\delta_0 \quad \text{in } \{u_h^e > \frac{1}{2}r^{\epsilon_0}\} \cap B_r.$$

Indeed, since u is a solution in the set $\{u > \varphi\}$ and a supersolution in B_1 , we conclude that the only points where the inequality in (5.11) can fail are those with $x + he \in \{u = \varphi\}$. At these points

$$u_h^e(x) \leq \varphi_h^e(x) \leq \delta_0 h^{\beta-\alpha} \quad (\text{or } \delta_0 r^{\beta-1} h^{1-\alpha} \quad \text{if } \beta > 1) \leq \frac{1}{2}r^{\epsilon_0}.$$

Moreover, call $K_T = \chi_{B_{1/4}} K$, then for a universal $c > 0$,

$$(5.12) \quad \mathcal{L}_{K_T} u_h^e \geq -c \quad \text{in } \{u_h^e > \frac{1}{2}r^{\epsilon_0}\} \cap B_r.$$

Indeed for x in such set $u_h^e(x) > 0$ and we have,

$$\mathcal{L}_{K_T} u_h^e \geq -\delta_0 - \int_{\mathcal{C}B_{1/4}(x)} u_h^e(y) K(y - x) dy.$$

Call the second term E . Then, one easily sees that

$$|E| \leq \frac{1}{h^\alpha} (E_1 + E_2 + E_3),$$

with

$$\begin{aligned} E_1 &:= \int_{A_1} |u(x+z)| |K(z) - K(z-he)| dz, \quad A_1 = \mathcal{C}(B_{1/4} \cup B_{1/4}(he)); \\ E_2 &:= \int_{A_2} |u(x+z)| K(z-he) dz, \quad A_2 := B_{1/4} \setminus B_{1/4}(he); \\ E_3 &:= \int_{A_3} |u(x+z)| K(z) dz, \quad A_3 := B_{1/4}(he) \setminus B_{1/4}. \end{aligned}$$

Since $h \leq r = 2^{-k}$ with k large, and u is bounded in B_1 , then $E_2, E_3 \leq Ch$. To bound E_1 we use that $\|u\|_{L^1(d\omega)} \leq \delta_0$ and assumption (5.1). We thus obtain $E_3 \leq Ch$ as well and by collecting all these bounds we obtain the desired claim.

Now, let

$$\tilde{u}(x) := r^{-(\alpha+\epsilon_0)} u(rx),$$

be the rescaling of u and notice that from $u \geq \varphi$ and (5.10) applied with $x = 0$, $he = ry$, $y \in B_1$ we find

$$(5.13) \quad -\delta_0 \leq \tilde{u}(y) \leq |y|^\alpha \quad \text{in } B_1.$$

Let $h \leq r/2$, and write $h = r\tilde{h}$, with $\tilde{h} \leq 1/2$. Then

$$v(x) := \tilde{u}_h^e(x) = r^{-\epsilon_0} u_h^e(rx),$$

is the rescaling of u_h^e from B_r to the unit ball, and from (5.10), (5.12) in B_1 we obtain that in B_1

$$-2 \leq v \leq 1, \quad \mathcal{L}_{\tilde{K}_T} v \geq -\delta_0 \quad \text{in } \{v > \tfrac{1}{2}\},$$

where the lower bound on v follows from (5.10) applied for $-e$. Here

$$\tilde{K}_T = \chi_{B_{1/4r}} K.$$

Now we claim that $|\{v < 1 - c\} \cap B_1| \geq c$ for some fixed c small universal. The reason is that if v is close to 1 in almost all B_1 then we contradict that $\tilde{u} \geq -\delta_0$. Indeed, assume for simplicity that $e = e_n$ and we integrate v in the cylinder

$$\mathcal{C} := \left\{ |x'| \leq \frac{1}{8}, \quad x_n \in \left[-\frac{3}{4}, \frac{1}{4}\right] \right\}.$$

For each segment in the e_n direction $l_{x'} = \{(x', x_n) | x_n \in [-\frac{3}{4}, \frac{1}{4}]\}$ of length 1 included in \mathcal{C} we have (see (5.13))

$$\begin{aligned} \int_{l_{x'}} v dx_n &= \tilde{h}^{-\alpha} \left(\int_{\frac{1}{4}}^{\frac{1}{4}+\tilde{h}} \tilde{u} dx_n - \int_{-\frac{3}{4}}^{-\frac{3}{4}+\tilde{h}} \tilde{u} dx_n \right) \\ &\leq \tilde{h}^{1-\alpha} \left(\left(\frac{7}{8}\right)^\alpha + \delta_0 \right) \leq 1 - c, \end{aligned}$$

and our claim follows.

Now the proof of diminish of oscillation for v follows as in Step 1. We remark that in bounding $\mathcal{L}_{\tilde{K}_T} \tilde{v}$ at the contact point, we will not have a term as I_3^2 , since the kernel \tilde{K} is truncated. All the other terms can be bounded with similar arguments as above.

In conclusion property (5.10) is proved and this implies that $u \leq r^{\alpha+\epsilon_0}$ in B_r for all dyadic balls, thus u is pointwise $C^{\alpha+\epsilon_0}$ at 0. Now we can extend as above the pointwise regularity from the set $\{u = \varphi\}$ to the whole $B_{1/4}$, and obtain the desired conclusion. \square

We show that when $s > \frac{1}{2}$, then the result of Proposition 5.6 can be improved.

Proposition 5.7. *Let u satisfy (5.2), (5.3) and assume $s > 1/2$,*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{\epsilon_0}(B_1)} \leq 1,$$

for some $\beta \neq 2s$. Then $u \in C^\alpha(B_1)$ for $\alpha = \min\{\beta, 2s + \epsilon_0\}$ with

$$\|u\|_{C^\alpha(B_{1/2})} \leq C.$$

Proof. Assume that $\|u\|_{L^1(d\omega)}, \|\varphi\|_{C^\beta}, \|f\|_{C^{\epsilon_0}}$ are all smaller than δ_0 , and assume also that $u(0) = \varphi(0) = 0$, and $\nabla\varphi(0) = 0$ if $\beta > 1$. We treat the case when $\beta \geq 2s + \epsilon_0$.

We prove by induction that there exists a sequence of radii $1 = r_1 > r_2 > \dots$ with $r_{k+1}/r_k \in [\rho_0, 1/2)$ for some fixed ρ_0 such that

$$(5.14) \quad \int |u|(\max\{r, |x|\})^{-(n+1+2s)} dx \leq r^{\epsilon_0-1}.$$

Assume that this holds for some $r = r_k$. We let

$$\tilde{u}(x) = r^{-2s-\epsilon_0} u(rx), \quad \tilde{\varphi}(x) = r^{-2s-\epsilon_0} \varphi(rx), \quad \tilde{f}(x) = r^{-\epsilon_0} f(rx),$$

and we have

$$\mathcal{L}_{\tilde{K}} \tilde{u} \leq \tilde{f} \quad \text{in } B_1, \quad \mathcal{L}_{\tilde{K}} \tilde{u} = \tilde{f} \quad \text{in } \{\tilde{u} > \tilde{\varphi}\} \cap B_1,$$

and

$$\text{osc}_{B_1} \tilde{f} \leq \delta_0, \quad |\tilde{\varphi}(x)| \leq \delta_0 |x|^{2s+\epsilon_0} \quad \text{in } B_1.$$

Moreover, (5.14) is equivalent to

$$(5.15) \quad \int |\tilde{u}|(\max\{1, |x|\})^{-(n+1+2s)} dx \leq 1.$$

We want to show that there exists $\rho \in [\rho_0, \frac{1}{2})$ such that

$$(5.16) \quad \int |\tilde{u}|(\max\{\rho, |x|\})^{-(n+1+2s)} dx \leq \rho^{\epsilon_0-1},$$

and then the induction hypothesis (5.14) is satisfied for $r_{k+1} = \rho r_k$.

Notice that $\tilde{u} + \delta_0$ satisfies the hypotheses of the Lemma 5.3 hence $\tilde{u} \leq C$ in $B_{1/2}$. Now we distinguish two cases.

Case 1: $\tilde{u} \leq \delta_0$ in $B_{1/4}$. Then (5.16) is satisfied clearly satisfied for $\rho = \rho_0$ small, provided that $\delta_0 \ll \rho_0$ is chosen sufficiently small.

Case 2: $\tilde{u} > \delta_0$ for some point in $B_{1/4}$. The according to Remark 5.4 we can slide a parabola of fixed opening by above and obtain a contact point in $\{\tilde{u} > \delta_0 > \tilde{\varphi}\}$ thus

$$\mathcal{L}_{\tilde{K}_T} \tilde{u}(0) \leq C.$$

Since $\tilde{\varphi}$ is tangent by below to \tilde{u} at 0 the above inequality implies

$$(5.17) \quad \int_{B_1} |\tilde{u}| |x|^{-n-2s} dx \leq C.$$

On the other hand, if we assume by contradiction that (5.16) holds in the opposite direction for all $\rho \in (\rho_0, 1/2)$ then we can integrate this inequality in ρ and obtain

$$\int |\tilde{u}|(\min\{1, |x|\}) |x|^{-(n+1+2s)} dx \geq \eta(\rho_0, \epsilon_0),$$

with $\eta(\rho_0, \epsilon_0) \rightarrow \infty$ as $\rho_0, \epsilon_0 \rightarrow 0$. This contradicts (5.15), (5.17) by choosing ϵ_0, ρ_0 sufficiently small.

In conclusion property (5.14) is proved, and from the argument above we obtain $u(x) \leq C|x|^{2s+\epsilon_0}$ in B_1 . This means that u is pointwise $C^{2s+\epsilon_0}$ in the set $\{u = \varphi\}$, and this can be extended to the whole $B_{1/2}$ as before.

When $\beta \in (2s, 2s + \epsilon_0)$ the argument above applies with ϵ_0 replaced by $\beta - 2s$.

Finally, when $\beta < 2s$ the proof is simpler. The rescaling $\tilde{u}(x) = r^{-\beta}u(rx)$ satisfies $\|\tilde{u}\|_{L^1(\mathbb{R}^n, d\omega)} \leq C$, (since now $\tilde{\varphi}$ is integrable at infinity) and we can apply Lemma 5.2 directly to obtain the pointwise C^β estimate at the origin. In this case we only require $f \in L^\infty$. \square

6. THE CASE OF THE FRACTIONAL LAPLACIAN: FREE BOUNDARY REGULARITY

In the special case when

$$K_1(y) = \frac{1}{|y|^{n+2s_1}}$$

the operator \mathcal{L}_{K_1} is the fractional Laplacian Δ^{s_1} and we obtain the optimal regularity of the minimizing pair in the two membranes problem, see Theorem 2.6. This improvement is due to the fact that the optimal $C^{1,s}$ regularity in the obstacle problem for the fractional Laplacian is known. Precisely, assume that u is a solution of the thin obstacle problem in B_1 with obstacle φ by below, that is u, φ are continuous in B_1 , $u \in L^1(\mathbb{R}^n, d\omega)$, and

$$(6.1) \quad u \geq \varphi \quad \text{in } B_1,$$

$$(6.2) \quad \Delta^s u \leq f \quad \text{in } B_1, \quad \text{and} \quad \Delta^s u = f \quad \text{in } \{u > \varphi\} \cap B_1.$$

The following result holds (see Section 1 for the notion of regular points.)

Theorem 6.1 (Optimal regularity). *Let u be a solution to (6.1), (6.2), with*

$$\|u\|_{L^1(\mathbb{R}^n, d\omega)}, \|\varphi\|_{C^\beta(B_1)}, \|f\|_{C^{\beta-2s}(B_1)} \leq 1, \quad \text{for some } \beta > 1 + s.$$

Then $u \in C^{1+s}(B_1)$ and

$$\|u\|_{C^{1+s}(B_{1/2})} \leq C.$$

Moreover, the free boundary $\Gamma := \partial\{u = \varphi\}$ is a $C^{1,\gamma}$ surface in a neighborhood of each of its regular points. The constants C, γ depend on n, s , and β .

Theorem 6.1 was obtained by Caffarelli, Salsa and Silvestre in [CSS]. The main tool in the proof is to establish a version of Almgren's frequency formula for the "extension" of u to \mathbb{R}^{n+1} . Theorem 6.1 is proved in [CSS] in the case when $\varphi \in C^{2,1}$ (i.e. $\beta = 3$). Below we show that the Almgren's monotonicity formula still holds when $\beta > 1 + s$. Since this is the only place in the proof in [CSS] where the regularity of the data is needed, we obtain the version of Theorem 6.1 above.

Finally we remark that in the case when $\beta \in (2s, 1 + s)$ the $C^{1,\alpha}$ regularity of u with $\alpha < \beta$ was obtained by Silvestre in [S2].

6.1. Almgren's monotonicity formula. In this section, \mathcal{B}_r will denote a ball in \mathbb{R}^{n+1} and $B_r := \mathcal{B}_r \cap \{x_{n+1} = 0\}$. Also, $X = (x, x_{n+1})$ is a point in \mathbb{R}^{n+1} and often we call $y = x_{n+1}$.

After subtracting an explicit function whose fractional Laplacian equals f , we may assume without loss of generality that $f = 0$. Let u be a solution in B_2 to the

thin obstacle problem

$$(6.3) \quad \begin{aligned} u &\geq \varphi && \text{in } B_2 \subset \mathbb{R}^n \\ \Delta^s u &= 0 && \text{in } \{u > \varphi\} \cap B_2 \\ \Delta^s u &\leq 0 && \text{in } B_2 \end{aligned}$$

with $\varphi : B_2 \rightarrow \mathbb{R}$ a continuous function.

Consider the equivalent (localized) problem obtained extending u to \mathbb{R}^{n+1} , evenly in the $y = x_{n+1}$ direction,

$$\begin{aligned} u(x, 0) &\geq \varphi \quad \text{for } x \in B_2 \\ u(x, y) &= u(x, -y) \\ L_a u &= \operatorname{div}(|y|^a \nabla u(x, y)) = 0 \quad \text{in } \mathcal{B}_2 \setminus \{u(x, 0) = \varphi(x)\} \\ L_a u &\leq 0 \quad \text{in } \mathcal{B}_2 \text{ in the distributional sense} \end{aligned}$$

where

$$a := 1 - 2s, \quad a \in (-1, 1).$$

Assume $\varphi \in C^{1,s+\delta}(B_2)$, for some $\delta > 0$ and $\|\varphi\|_{C^{s+\delta}} \leq 1$. We extend φ to \mathcal{B}_1 in the following way:

$$(6.4) \quad \tilde{\varphi}(x, y) := \varphi * \rho_{|y|},$$

with $\rho_r(X) := r^{-n-1} \rho(X/r)$, and ρ a symmetric mollifier supported in \mathcal{B}_1 . Then it is easy to check that $\tilde{\varphi} \in C^{1,s+\delta}$ is even in y and is smooth away from $\{y = 0\}$, and

$$(6.5) \quad \|D^2 \tilde{\varphi}\| \leq C|y|^{s+\delta-1} \Rightarrow |y|^{-a} L_a \tilde{\varphi} \leq C|y|^{s+\delta-1}.$$

Define,

$$\tilde{u}(x, y) = u(x, y) - \tilde{\varphi}(x, y),$$

and let $\Lambda := \{\tilde{u}(x, 0) = 0\}$. Then \tilde{u} satisfies

$$\begin{cases} \tilde{u}(x, 0) \geq 0 & \text{for } x \in B_1 \\ \tilde{u}(x, y) = \tilde{u}(x, -y) \\ L_a \tilde{u} = -L_a \tilde{\varphi} & \text{in } \mathcal{B}_1 \setminus \Lambda \end{cases}$$

Denote by

$$F(r) := \frac{1}{r^{n+a}} \int_{\partial \mathcal{B}_r} \tilde{u}^2 |y|^a d\sigma,$$

and notice that if for example \tilde{u} is homogenous of degree σ , then $F(r) = cr^{2\sigma}$, hence $\frac{1}{2}r \frac{d}{dr} \log F = \sigma$.

Theorem 6.2 (Almgren's monotonicity formula). *Let $0 \in \Lambda$ and $\alpha \in (s, s + \delta)$. There exist constants C_0 and r_0 depending on α , s , n , and δ such that the function*

$$\Phi_{\tilde{u}}(r) := \frac{1}{2}(r + C_0 r^{1+\epsilon}) \frac{d}{dr} \log \left(\max\{F(r), r^{2(1+\alpha)}\} \right)$$

is monotone increasing for all $0 < r \leq r_0$, where $\epsilon > 0$ is small so that $s + \delta \geq \alpha + \epsilon$.

For simplicity we also use the notation of the ‘‘averages’’ of a function g with respect to the measures $|y|^a d\sigma$ and $|y|^a dX$:

$$\oint_{\partial \mathcal{B}_r} g |y|^a d\sigma := \frac{1}{r^{n+a}} \int_{\partial \mathcal{B}_r} g |y|^a d\sigma$$

and

$$\oint_{\mathcal{B}_r} g |y|^a dX := \frac{1}{r^{n+1+a}} \int_{\mathcal{B}_r} g |y|^a dX.$$

With this notation,

$$F(r) := \oint_{\partial \mathcal{B}_r} \tilde{u}^2 |y|^a d\sigma,$$

and

$$F'(r) = 2 \oint_{\partial \mathcal{B}_r} \tilde{u} \tilde{u}_\nu |y|^a d\sigma.$$

First, we prove the following preliminary lemma.

Lemma 6.3. *Assume $F(r) \geq r^{2(1+\alpha)}$. Then, for r small*

$$\begin{aligned} \oint_{\mathcal{B}_r} \tilde{u}^2 |y|^a dX &\leq CF(r). \\ r^{-1} F'(r) &\sim \oint_{\mathcal{B}_r} |\nabla \tilde{u}|^2 |y|^a dX \geq Cr^{-2} F(r) \end{aligned}$$

Proof. Assume for simplicity that $u(0) = \varphi(0) = 0$, $\nabla \varphi(0) = 0$, hence

$$|\tilde{\varphi}| \leq Cr^{1+s+\delta} \leq r^{1+\alpha+\epsilon} \quad \text{in } B_r,$$

hence the functions u and \tilde{u} are “the same” up to an error of $r^{1+\epsilon}$. Since $F(r) \geq r^{2(1+\alpha)}$ we obtain

$$\oint_{\partial \mathcal{B}_r} u^2 |y|^a d\sigma \sim \oint_{\partial \mathcal{B}_r} \tilde{u}^2 |y|^a d\sigma = F(r).$$

Since $L_a u = 0$ in the set $\{|u| > r^{1+\alpha+\epsilon}\}$ we may apply the mean value inequality for the L_a -subharmonic function

$$((|u| - r^{1+\alpha+\epsilon})^+)^2$$

and obtain that its average in \mathcal{B}_r is bounded by its average on $\partial \mathcal{B}_r$. This easily gives the first inequality above.

For the second inequality we have $L_a u \leq 0$ and $u(0) = 0$, hence the average of u on $\partial \mathcal{B}_r$ is negative. From this and the version of Poincaré inequality written for $\partial \mathcal{B}_r$ (see Lemma 2.10 in [CSS]) we obtain

$$r^2 \oint_{\mathcal{B}_r} |\nabla u|^2 |y|^a dX \geq c \oint_{\partial \mathcal{B}_r} (u^+)^2 |y|^a d\sigma.$$

Moreover, similarly to the quoted lemma, since a function v in the weighted Sobolev space $W^{2,1}(\mathcal{B}_1, |y|^a)$ has trace in $L^2(B_1)$, we also have the following version of Poincaré inequality:

$$r^2 \oint_{\mathcal{B}_r^+} |\nabla v|^2 |y|^a dX \geq c \oint_{\partial \mathcal{B}_r^+} (v - \bar{v})^2 |y|^a d\sigma$$

with

$$\bar{v} := \oint_{B_r} v(x, 0) dx.$$

Hence, since $u \geq -r^{1+\alpha+\epsilon}$ on B_r , we deduce that

$$r^2 \oint_{\mathcal{B}_r} |\nabla u|^2 |y|^a dX \geq c \oint_{\partial \mathcal{B}_r} (u^-)^2 |y|^a dX - Cr^{2(1+\alpha+\epsilon)}.$$

Using that $\nabla \tilde{u} = \nabla u + O(r^{1+\alpha+\epsilon})$ we obtain

$$\int_{\mathcal{B}_r} |\nabla \tilde{u}|^2 |y|^a dX \geq Cr^{-2} F(r).$$

Finally,

$$\int_{\mathcal{B}_r} (\tilde{u} L_a \tilde{u} + |\nabla \tilde{u}|^2 |y|^a) dX = \int_{\mathcal{B}_r} \operatorname{div}(|y|^a \tilde{u} \nabla \tilde{u}) dX = \int_{\partial \mathcal{B}_r} \tilde{u} \tilde{u}_\nu |y|^a d\sigma,$$

thus, since $\tilde{u} L_a \tilde{u} = -\tilde{u} L_a \tilde{\varphi}$ we have

$$\frac{1}{2r} F'(r) = \frac{1}{r} \int_{\partial \mathcal{B}_r} \tilde{u} \tilde{u}_\nu |y|^a d\sigma = \int_{\mathcal{B}_r} (|\nabla \tilde{u}|^2 - |y|^{-a} \tilde{u} L_a \tilde{\varphi}) |y|^a dX.$$

By Cauchy-Schwartz and the property (6.5) of $\tilde{\varphi}$ we have

$$\begin{aligned} \left| \int_{\mathcal{B}_r} \tilde{u} (|y|^{-a} L_a \tilde{\varphi}) |y|^a d\sigma \right| &\leq \left(\int_{\mathcal{B}_r} \tilde{u}^2 |y|^a d\sigma \right)^{1/2} \left(\int_{\mathcal{B}_r} (|y|^{-a} L_a \tilde{\varphi})^2 |y|^a d\sigma \right)^{1/2} \\ &\leq Cr^{\alpha+\epsilon-1} F(r)^{1/2}, \end{aligned}$$

and we obtain the desired conclusion (using also that $F(r) \geq r^{2(1+\alpha)}$). \square

Proof of Theorem 6.2. It is enough to consider the case when

$$F(r) \geq r^{2(1+\alpha)}.$$

Then,

$$\Phi_{\tilde{u}}(r) = \frac{1}{2} (r + C_0 r^{1+\epsilon}) \frac{F'(r)}{F(r)}.$$

We compute its logarithmic derivative and show that it is non-negative. Precisely, we look at the quantity:

$$N(r) := \frac{1}{r} + \frac{\epsilon C_0 r^{\epsilon-1}}{1 + C_0 r^\epsilon} + \frac{F''(r)}{F'(r)} - \frac{F'(r)}{F(r)}.$$

As in Lemma 6.3,

$$(6.6) \quad \int_{\partial \mathcal{B}_r} \tilde{u} \tilde{u}_\nu |y|^a d\sigma = \int_{\mathcal{B}_r} (|\nabla \tilde{u}|^2 + |y|^{-a} \tilde{u} L_a \tilde{u}) |y|^a dX.$$

Thus,

$$F''(r) = -\frac{(n+a)}{r} F'(r) + 2 \int_{\partial \mathcal{B}_r} (|\nabla \tilde{u}|^2 + |y|^{-a} \tilde{u} L_a \tilde{\varphi}) |y|^a d\sigma.$$

As in [CSS] we can estimate that

$$\begin{aligned} \int_{\partial \mathcal{B}_r} |\nabla \tilde{u}|^2 |y|^a d\sigma &= 2 \int_{\partial \mathcal{B}_r} (\tilde{u}_\nu)^2 |y|^a d\sigma + \frac{n+a-1}{r} \int_{\partial \mathcal{B}_r} \tilde{u} \tilde{u}_\nu |y|^a d\sigma \\ &\quad - \int_{\mathcal{B}_r} ((n+a-1)\tilde{u} - 2X \cdot \nabla \tilde{u}) (|y|^{-a} L_a \tilde{\varphi}) |y|^a dX. \end{aligned}$$

Hence,

$$N(r) = \frac{\epsilon C_0 r^{\epsilon-1}}{1 + C_0 r^\epsilon} + \frac{4 \int_{\partial \mathcal{B}_r} (\tilde{u}_\nu)^2 |y|^a d\sigma}{F'(r)} - \frac{F'(r)}{F(r)} + \frac{H(r)}{F'(r)},$$

with

$$\begin{aligned} H(r) &= 2 \int_{\partial \mathcal{B}_r} \tilde{u}(|y|^{-a} L_a \tilde{\varphi}) |y|^a d\sigma - (n+a-1) \int_{\mathcal{B}_r} \tilde{u}(|y|^{-a} L_a \tilde{\varphi}) |y|^a dX \\ &\quad + 4 \int_{\mathcal{B}_r} (X \cdot \nabla \tilde{u}) (|y|^{-a} L_a \tilde{\varphi}) |y|^a dX \\ &:= H_1(r) + H_2(r) + H_3(r). \end{aligned}$$

By Cauchy-Schwartz, we conclude that (for r small)

$$(6.7) \quad N(r) \geq \frac{\epsilon C_0 r^{\epsilon-1}}{1 + C_0 r^\epsilon} + \frac{H(r)}{F'(r)} \geq \epsilon \frac{C_0}{2} r^{\epsilon-1} + \frac{H(r)}{F'(r)}.$$

We now estimate $H(r)$. As in Lemma 6.3 we use property (6.5) of $\tilde{\varphi}$ and conclude

$$\left| \int_{\mathcal{B}_r} \tilde{u}(|y|^{-a} L_a \tilde{\varphi}) |y|^a dX \right| \leq C r^{\alpha+\epsilon-1} F(r)^{1/2},$$

and with a similar computation

$$\left| \int_{\partial \mathcal{B}_r} \tilde{u}(|y|^{-a} L_a \tilde{\varphi}) |y|^a d\sigma \right| \leq C r^{\alpha+\epsilon-1} F(r)^{1/2}.$$

In the same way,

$$\begin{aligned} \left| \int_{\mathcal{B}_r} (X \cdot \nabla \tilde{u}) (|y|^{-a} L_a \tilde{\varphi}) |y|^a dX \right| &\leq r \left(\int_{\mathcal{B}_r} |\nabla \tilde{u}|^2 |y|^a dX \right)^{1/2} \left(\int_{\mathcal{B}_r} (|y|^{-a} L_a \tilde{\varphi})^2 |y|^a dX \right)^{1/2} \\ &\leq r^{\alpha+\epsilon} \left(\int_{\mathcal{B}_r} |\nabla \tilde{u}|^2 |y|^a dX \right)^{1/2}, \end{aligned}$$

hence by Lemma 6.3

$$\frac{|H_1(r)|}{F'(r)} \leq C r^{\epsilon-1}, \quad \frac{|H_2(r)|}{F'(r)} \leq C r^{\epsilon-1}, \quad \frac{|H_3(r)|}{F'(r)} \leq C r^{\epsilon-1}.$$

Combining these estimates with (6.7) we get that $N(r) > 0$ for C_0 large and r small. \square

Now the arguments in [CSS] apply, and they give that if $0 \in \partial \Lambda$ then the limit $\Phi(0+)$ can take only two values: $1+s$ and $1+\alpha$, and this implies the $C^{1,s}$ regularity of u . If this limit $\Phi(0+)$ equals $1+s$ we say that 0 is a regular point. Then the monotonicity formula allows us to perform the blow-up analysis at a regular point and to obtain the $C^{1,\gamma}$ regularity of the free boundary. In view of this, we sharpen the regularity results of [CSS] for the thin obstacle problem, in the case when the obstacle $\varphi \in C^{1,s+\delta}$, and obtain Theorem 6.1.

6.2. An extension of Theorem 6.2. We consider here the case when the obstacle φ is $C^{1+s+\delta}$ only in a certain pointwise sense and u has nearly optimal regularity. This case appears in [CDS] where we deal with the obstacle problem for non-local minimal surfaces. Precisely, we obtain the following proposition.

Proposition 6.4. *Let $u \in C^{2s+\epsilon}$ solve the obstacle problem (6.1)-(6.2), $0 \in \partial \Lambda$. Assume that $\|u\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$ and ∇u is pointwise $C^{s-\frac{\delta}{2}}$ at the origin, i.e.*

$$(6.8) \quad |\nabla u(x)| \leq |x|^{s-\frac{\delta}{2}} \quad \text{in } B_1.$$

If $\varphi \in C^{2s+\epsilon}$, $\nabla\varphi$ is pointwise $C^{s+\delta}$ at the origin i.e., for all $r < 1$

$$(6.9) \quad \begin{aligned} |\nabla\varphi|_{L^\infty(B_r)} &\leq r^{s+\delta} \quad \text{if } s \in (0, \tfrac{1}{2}) \\ [\nabla\varphi]_{C^{2s+\delta-1}(B_r)} &\leq r^{1-s} \quad \text{if } s \in [\tfrac{1}{2}, 1), \end{aligned}$$

and f satisfies

$$(6.10) \quad \begin{aligned} [f]_{C^\gamma(B_r)} &\leq Cr^{s+\delta} \quad \text{for some } \gamma > 1 - 2s, \text{ if } s \in (0, 1/2), \\ [f]_{C^\delta(B_r)} &\leq Cr^{1-s} \quad \text{if } s \in [1/2, 1), \end{aligned}$$

then u is pointwise $C^{1,s}$ at the origin i.e.

$$(6.11) \quad |u(x)| \leq C|x|^{1+s} \quad \text{in } B_1,$$

for some C depending only on n, s and δ .

The Proposition above will follow if we show that the monotonicity formula can be applied under these hypotheses.

Assume first that the right hand side f equals 0. Since $u, \varphi \in C^{2s+\epsilon}$ in B_1 , the integrations by parts performed in the monotonicity formula are justified. Now, using the boundary estimates for the equation $L_a u = 0$ together with $y^a u_y(0, y) \rightarrow 0$ as $y \rightarrow 0$ which is a consequence of $0 \in \partial\Lambda$, we find that the extension $u(X)$ satisfies in \mathcal{B}_r

$$(6.12) \quad |u| \leq Cr^{1+s-\frac{\delta}{2}}, \quad |X \cdot \nabla u| \leq Cr^{1+s-\frac{\delta}{2}}.$$

In view of (6.9), the extension $\tilde{\varphi}$ defined in (6.4) satisfies in \mathcal{B}_r

$$|\tilde{\varphi}| \leq r^{s+\delta+1}, \quad |\nabla\tilde{\varphi}| \leq r^{s+\delta},$$

and

$$\begin{aligned} \frac{|u_y|}{|y|}, |D^2\tilde{\varphi}| &\leq Cr^{s+\delta}|y|^{-1} \quad \text{if } s \in (0, \tfrac{1}{2}) \text{ or,} \\ \frac{|u_y|}{|y|}, |D^2\tilde{\varphi}| &\leq r^{1-s}|y|^{2s+\delta-2} \quad \text{if } s \in [\tfrac{1}{2}, 1). \end{aligned}$$

Since $a = 1 - 2s$ and

$$|y|^{-a}|L_a\tilde{\varphi}| \leq C \left(|D^2\tilde{\varphi}| + \frac{|u_y|}{|y|} \right),$$

we see that $|y|^{-a}L_a\tilde{\varphi}$ is integrable with respect to the measures $|y|^a dX$ and $|y|^a d\sigma$, and its averages with respect to these measures in \mathcal{B}_r , respectively $\partial\mathcal{B}_r$ are bounded by $Cr^{s+\delta-1}$.

From these inequalities we see that $\tilde{u} = u - \tilde{\varphi}$ satisfies the same bounds in (6.12) and we can estimate the error terms H_1, H_2, H_3 by

$$Cr^{1+s-\frac{\delta}{2}}r^{s+\delta-1} = Cr^{2s+\delta/2} \leq Cr^{2\alpha+\epsilon},$$

provided that α is taken sufficiently close to s and $\epsilon > 0$ is small. The difference is that now we used the $L^\infty L^1$ bound for the product between the \tilde{u} terms and $|y|^{-a}L_a\tilde{\varphi}$ terms instead of the $L^2 L^2$ as before.

In the general case when the right hand side f is not 0, then the potential whose fractional Laplacian equals f must satisfy (6.9) and we need to impose the conditions in (6.10).

We mention that similar arguments with the ones that we provide above were used by Guillen in [G] in a slightly different context.

7. APPENDIX

Below we discuss the Schauder estimates for translation invariant integro-differential equations of the type

$$\mathcal{L}_K v(x) = P.V. \int (v(x+y) - v(x))K(y)dy,$$

with kernels K that satisfy

$$(7.1) \quad \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \quad 0 < \lambda \leq \Lambda,$$

$$(7.2) \quad |\nabla K(y)| \leq \Lambda |y|^{-(n+1+2s)}.$$

For convenience we state again the Schauder estimates used in Section 5.

Proposition 7.1. *Let K be a symmetric kernel that satisfies (7.1), and assume that $v \in L^1(\mathbb{R}^n, d\omega)$ satisfies*

$$\mathcal{L}_K v = f \quad \text{in } B_1, \quad \|v\|_{L^\infty(B_1)} \leq 1.$$

a) *If $\|f\|_{L^\infty(B_1)} \leq 1$, $\|v\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$ then*

$$\|v\|_{C^\alpha(B_{1/2})} \leq C(\alpha), \quad \text{for any } \alpha < 2s.$$

b) *If K satisfies (7.2) and*

$$\int_{CB_1} v |x|^{-(n+2s+1)} dx \leq 1, \quad [f]_{C^\gamma(B_1)} \leq 1, \quad \text{for some } \gamma \in (0, 1)$$

then

$$\|v\|_{C^{2s+\gamma}(B_{1/2})} \leq C(\gamma),$$

provided that $2s + \gamma$ is not an integer.

c) *Conversely, if K satisfies (7.2) and $\|v\|_{L^1(\mathbb{R}^n, d\omega)} \leq 1$, $\|v\|_{C^{2s+\gamma}(B_1)} \leq 1$, then*

$$\|f\|_{C^\gamma(B_{1/2})} \leq C.$$

We remark that the constant $C(\gamma)$ in part b) is independent on $\|f\|_{L^\infty}$ and $\|v\|_{L^1(\mathbb{R}^n, d\omega)}$.

We point out that by the results in [Ser], one could in fact relax the assumption (7.2) and require that it is satisfied only outside of a neighborhood of the origin.

We sketch the main steps in the proofs of parts a) and b) and use similar ideas as in Section 5. The proof of part c) is standard and we do not include it here.

First we obtain a Liouville type result for global solutions which have integrable decay at infinity.

Lemma 7.2. *The only global solutions to the equation*

$$\mathcal{L}_K v = 0 \quad \text{in } \mathbb{R}^n, \quad \|v\|_{L^\infty(B_{R_k})} \leq R_k^\alpha, \quad \text{with } R_k = 2^k, \quad k \geq 0,$$

for some $\alpha < 2s$, are constant if $s \leq \frac{1}{2}$, or linear if $s \in (\frac{1}{2}, 1)$

Proof. Since $\alpha < 2s$ we can apply the Hölder estimates from [S2] (as in Section 5) and we obtain that

$$(7.3) \quad \|v\|_{C^{\epsilon_0}(B_{1/2})} \leq C,$$

for some C, ϵ_0 depending only on n, s, α . Since the function $R_k^{-\alpha}v(R_kx)$ satisfies the same hypotheses as v , we can apply the estimate above for this function and obtain

$$(7.4) \quad \|v\|_{C^{\epsilon_0}(B_{R_k/2})} \leq CR_k^{\alpha-\epsilon_0}.$$

This means that the discrete difference function

$$\tilde{v} := \frac{1}{C_0} \frac{u(x+he) - u(x)}{h^{\epsilon_0}}, \quad |e| = 1, h \in [0, 1],$$

also satisfies the hypotheses of v with α replaced by $\alpha - \epsilon_0$.

We apply the estimates (7.4) for \tilde{v} and we obtain (see Lemma 5.6 in [CC])

$$\|v\|_{C^{2\epsilon_0}(B_{R_k})} \leq CR_k^{\alpha-2\epsilon_0}.$$

We iterate this result and distinguish 2 cases, if $\alpha < 1$ or $\alpha \geq 1$.

If $\alpha < 1$ then we find

$$\|v\|_{C^{\alpha'}(B_{R/2})} \leq CR^{\alpha-\alpha'},$$

for some $\alpha' \in (\alpha, 1)$ and by letting $R \rightarrow \infty$ we obtain that v is a constant.

If $\alpha \geq 1$ then we obtain

$$\|v\|_{C^{0,1}(B_{R_k})} \leq CR_k^{\alpha-1},$$

hence the discrete difference quotient $(v(x+he) - v(x))/h$ satisfies the hypotheses of the lemma with exponent $\alpha - 1 < 1$ thus it must be constant, which gives that v is a linear function. \square

Using compactness and Lemma 7.2 we obtain the following interior estimate.

Lemma 7.3. *Let w be a solution to the truncated kernel equation*

$$(7.5) \quad \mathcal{L}_{K_T} w = g \quad \text{in } B_{1/2}, \quad K_T := \chi_{B_{1/2}} K,$$

$$\|g\|_{L^\infty(B_{1/2})} \leq 1, \quad \|w\|_{L^\infty(B_1)} \leq 1.$$

Then, for any $\alpha < 2s$ we have

$$\|w\|_{C^\alpha(B_{1/4})} \leq C(\alpha).$$

Proof. We may assume that $\alpha \neq 1$. We need to show that if w satisfies

$$(7.6) \quad |w - l_k| \leq r_k^\alpha \quad \text{in } B_{r_k}, \quad r_k = 2^{-k},$$

for $k = 0, 1, \dots, m$ for some $m \geq k_0$ sufficiently large, then the inequality above holds also for $k = m + 1$. Here l_k is either a constant (for $\alpha < 1$) or a linear function (for $\alpha > 1$). Indeed, as $k_0 \rightarrow \infty$, we may find a subsequence of rescalings

$$\tilde{w} := r^{-\alpha} w(rx) \quad r = r_m$$

which converges uniformly on compact sets to a function v that satisfies the hypotheses of Lemma 7.2, and then (7.6) is clearly verified for k large. The uniform convergence on compact sets is once more guaranteed by Harnack inequality since \tilde{w} satisfies

$$\mathcal{L}_{\tilde{K}_T} \tilde{w} = \tilde{g}(x) := r^{2s-\alpha} g(rx), \quad \tilde{K}_T = \tilde{K} \chi_{B_{r^{-1}/2}},$$

and, as $k_0 \rightarrow \infty$, we have $\tilde{g} \rightarrow 0$ uniformly on compact sets. \square

The estimate in part a) of Proposition 7.1 follows from Lemma 7.3. We write the original equation in terms of the truncated kernel K_T and obtain

$$\mathcal{L}_{K_T} v(x) = f(x) - h(x) \quad \text{in } B_{1/2},$$

with

$$h(x) = \int_{\mathcal{C}B_{1/2}} (v(x+y) - v(x))K(y)dy,$$

and clearly

$$|h(x)| \leq C(\|v\|_{L^1(d\omega)} + |v(x)|) \leq C.$$

Next we apply Lemma 7.3 for difference quotients and obtain the $C^{2s+\gamma}, \gamma \in (0, 1)$, estimate.

Lemma 7.4. *Assume that K satisfies (5.1) (only outside a neighborhood of the origin) and w satisfies*

$$\mathcal{L}_{K_T} w = g + a w \quad \text{in } B_{1/2}, \quad \|w\|_{L^\infty(B_1)} \leq 1,$$

for some constant a with $|a| \leq 1$, and with

$$(7.7) \quad \|g\|_{C^{0,1}(B_{1/2})} \leq 1.$$

Then, if $\alpha < 2s$ we have

$$\|w\|_{C^{1+\alpha}(B_{1/4})} \leq C(\alpha).$$

Proof. Since the right hand side is bounded, we obtain by Lemma 7.3 a C^{α_0} bound for w in $B_{1/4}$ for some $\alpha_0 \in (0, 2s)$. Then we iterate Lemma 7.3 a finite number of times for the discrete differences of w and successively estimate w in $C^{\alpha_k}(B_{r_k})$ with $r_k = 4^{-k}$ and $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_m = 1$. Then we iterate this argument one more time and obtain the desired conclusion.

Notice that in order to apply Lemma 7.3 in B_{r_k} instead of B_1 we need to write the equation for the truncated kernel

$$K_{T,k} := K_T \chi_{B_{r_k/2}}.$$

Then the right hand side gets modified as follows

$$\mathcal{L}_{K_{T,k}} w(x) = g(x) - h_1(x) + h_2(x)$$

with

$$\begin{aligned} h_1(x) &= \int_{B_{1/2} \setminus B_{r_k/2}} w(x+y)K(y)dy = \int_{B_{1/2}(x) \setminus B_{r_k/2}(x)} w(y)K(x-y)dy, \\ h_2(x) &= aw(x) + \int_{B_{1/2} \setminus B_{r_k/2}} w(x)K(y)dy = (a + C(K))w(x). \end{aligned}$$

From our hypothesis on K , arguing as in Step 2 of Proposition 5.6, we find $\|h_1\|_{C^{0,1}} \leq C$. Since $\|h_2\|_{C^{\alpha_k}} \leq C\|w\|_{C^{\alpha_k}}$ in $B_{r_k/2}$, we can apply Lemma 7.3 for the discrete difference

$$\frac{w(x+he) - w(x)}{h^{\alpha_k}},$$

and obtain the $C^{\alpha_{k+1}}$ bound for w in $B_{r_k/4}$. □

Finally we prove part b) of Proposition 7.1.

Lemma 7.5. *Assume that v satisfies the hypotheses of part b) in Proposition 7.1 with*

$$\|v\|_{L^\infty(B_1)} \leq \delta_0, \quad [f]_{C^\gamma(B_1)} \leq \delta_0$$

for some small δ_0 . Then there exist polynomials p_k of degree $[\beta]$, and $p_0 \equiv 0$, such that

$$|v - p_k| \leq r_k^\beta \quad \text{in } B_{r_k}, \quad r_k = 2^{-k}, \quad \beta := 2s + \gamma,$$

for all $k \geq 0$.

Proof. We prove the lemma by induction by showing that if the conclusion holds up to some k large, then it holds also for $k + m_0$ for some fixed m_0 .

By the induction hypothesis, the coefficients of the polynomials p_k are uniformly bounded. Hence, if ψ is a cutoff function which is 1 in $B_{1/2}$ and 0 outside B_1 , then $p_k\psi$ is a C_0^∞ function with a uniform L^∞ bound and

$$\mathcal{L}_K(p_k\psi) = q \quad \text{with} \quad \|q\|_{C^{0,1}} \leq C.$$

Now we write the equation for the rescaling \tilde{v} of $v - p_k\psi$

$$\tilde{v}(x) = r^{-\beta}(v - p_k\psi)(rx), \quad r = r_k,$$

and obtain

$$\mathcal{L}_{\tilde{K}}\tilde{v}(x) = \tilde{g}(x) := r^{-\gamma}f(rx) + r^{-\gamma}q(rx) \quad \text{in } B_{r^{-1}}.$$

Notice that $[\tilde{g}]_{C^\gamma} \leq C\delta_0$ in B_2 provided that r is sufficiently small, and by the induction hypothesis

$$|\tilde{v}|_{L^\infty(B_1)} \leq 1, \quad |\tilde{v}(x)| \leq C|x|^\beta \quad \text{in } B_{r^{-1}} \setminus B_1,$$

which gives

$$(7.8) \quad \int_{CB_1} |\tilde{v}| |x|^{-(n+2s+1)} dx \leq C_0,$$

for a fixed C_0 depending only on γ and the universal constants.

As in Lemma 7.4 we write the equation for \tilde{v} in $B_{1/2}$ using the truncated kernel \tilde{K}_T and obtain

$$\mathcal{L}_{\tilde{K}_T}\tilde{v} = \tilde{g} - h + C(K)\tilde{v} =: g_0 + a \tilde{v},$$

with

$$h(x) = \int_{CB_{1/2}(x)} \tilde{v}(y)\tilde{K}(x-y)dy.$$

From the hypothesis on K and (7.8) we find

$$[h]_{C^{0,1}(B_2)} \leq C_1.$$

We use the estimate on the C^γ seminorm of g_0 and deduce that

$$(7.9) \quad \|g_0\|_{C^\gamma(B_{1/2})} \leq C,$$

by obtaining an L^∞ bound for g_0 . We achieve this by sliding the paraboloid $4|x|^2$ by above till it touches the graph of \tilde{v} at some $x_0 \in B_1$. Then $\mathcal{L}_{\tilde{K}_T}v(x_0) \leq C$ hence $g_0(x_0) \leq C$, and similarly we find a point x_1 such that $g_0(x_1) \geq -C$, and this proves (7.9).

By Lemma 7.3 the function \tilde{v} is uniformly Hölder continuous in B_1 . Moreover, g_0 is the sum of a Lipschitz function (with bounded Lipschitz norm) and a function with C^γ norm bounded by $C\delta_0$. By compactness and Lemma 7.4 we find that as

$\delta_0 \rightarrow 0$ we can approximate \tilde{v} uniformly in B_1 by a function with bounded $C^{1+\alpha}$ norm in $B_{1/4}$ (with $1 + 2s > 1 + \alpha > \beta$.) Thus we can find m_0 universal such that

$$|\tilde{v} - \tilde{p}| \leq \rho^\beta \quad \text{in } B_\rho, \quad \rho = 2^{-m_0}.$$

This means that the induction hypothesis holds for $k + m_0$, and the lemma is proved. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712, USA
E-mail address: `caffarel@math.utexas.edu`

DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE, COLUMBIA UNIVERSITY, NEW YORK, NY 10027, USA
E-mail address: `desilva@math.columbia.edu`

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027, USA
E-mail address: `savin@math.columbia.edu`